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PART I

THE OPEN UNIVERSITY

Mathematics: A Third Level Course

PART I CALCULUS ON EUCLIDEAN SPACE

DIFFERENTIAL GEOMETRY

THE OPEN UNIVERSITY

Mathematics: A Third Level Course

O.U. COURSE UNITS COLLECTION
FOR REFERENCE ONLY

DIFFERENTIAL GEOMETRY PART I

CALCULUS ON EUCLIDEAN SPACE

A commentary on Chapter I of O'Neill's
Elementary Differential Geometry
Prepared by the Course Team

THE OPEN UNIVERSITY PRESS

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1.1

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Set Book

Barrett O'Neill: *Elementary Differential Geometry* (Academic Press, 1966). It is essential to have this book: the course is based on it and will not make sense without it.

Bibliography

The set books for M201, M231 and MST 282 are referred to occasionally; they are useful but not essential. They are:

D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins: *An Introduction to Linear Analysis* (Addison-Wesley, 1966).

E.D. Nering: *Linear Algebra and Matrix Theory* (John Wiley, 1970).

M. Spivak: *Calculus,* paperback edition (W.A. Benjamin/Addision-Wesley, 1973).

R.C. Smith and P. Smith: *Mechanics,* SI Edition (John Wiley, 1972).

Conventions

Before starting work on this text, please read M334 Part Zero. Consult the Errata List and the Stop Press and make any necessary alterations for this chapter in the set book.

Unreferenced pages and sections denote the set book. Otherwise

O'Neill	denotes the set book;
Text	denotes the correspondence text;
KKOP	denotes *An Introduction to Linear Analysis* by D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins;
Nering	denotes *Linear Algebra and Matrix Theory* by E.D. Nering;
Spivak	denotes *Calculus* by M. Spivak;
Smith	denotes *Mechanics* by R.C. Smith and P. Smith.

Reference to Open University Courses in Mathematics take the form:

Unit M100 22, Linear Algebra I

Unit MST 281 10, Taylor Approximation

Unit M201 16, Euclidean Spaces I: Inner Products

Unit M231 2, Functions and Graphs

Unit MST 282 1, Some Basic Tools.

I.1 INTRODUCTION AND EUCLIDEAN SPACE

Introduction

In this section we introduce some notation that will be used throughout the course. We review *vector spaces* and *partial differentiation.* In the additional text we give some rules for the evaluation of partial derivatives.

READ: Introduction and Section I.1 (pages 1-5).

Comments

(i) *Page 1: line -1* In the correspondence text we shall sometimes write $g \circ f$ for the composite function.

(ii) *Page 4: Definition 1.2* From now on we shall use x, y, z in this way only. We shall not use x as a typical element of the real line, or indeed any set as O'Neill does on pages 1 and 2 of the Introduction.

Additional Text

Partial differentiation Suppose $p = (p_1, p_2, p_3)$ is a point of E^3 and $f : E^3 \longrightarrow R$ is a differentiable function. To obtain $(\partial f/\partial x_1)(p)$ we first construct the function $F : R \longrightarrow R$ such that $F(t) = f(t, p_2, p_3)$ and then $(\partial f/\partial x_1)(p) = F'(p_1)$. That is we consider x_2 and x_3 as constants and differentiate in the usual way. Partial differentiation satisfies the following rules. Let a, b be real numbers; f, g differentiable functions from E^3 to R and h a differentiable function from R to R, then

(1) $$\frac{\partial(af+bg)}{\partial x_i} = a\frac{\partial f}{\partial x_i} + b\frac{\partial g}{\partial x_i} \qquad (\text{for } i = 1, 2, 3, \ldots)$$

(2) $$\frac{\partial(fg)}{\partial x_i} = \frac{\partial f}{\partial x_i}\, g + f\frac{\partial g}{\partial x_i} \qquad (\text{for } i = 1, 2, 3, \ldots)$$

(3) $$\frac{\partial(h(f))}{\partial x_i} = h'(f)\frac{\partial f}{\partial x_i} \qquad (\text{for } i = 1, 2, 3, \ldots)$$

Equation (3) may be more recognizable if the composite functions are written as $h \circ f$ and $h' \circ f$. All these results follow from the similar rules for ordinary differentiation. Lists of useful derivatives are to be found in

M231 Handbook, page 45 *et seq.*

MST 282 Handbook, page 27.

Example Find $\dfrac{\partial(2x + xz\sin(x^2y))}{\partial x}$.

We have

$$\frac{\partial(2x + xz\sin(x^2y))}{\partial x} = 2\frac{\partial x}{\partial x} + \frac{\partial(xz\sin(x^2y))}{\partial x}, \quad \text{by result (1)}$$

$$= 2 + \frac{\partial(xz)}{\partial x}\sin(x^2y) + xz\frac{\partial(\sin(x^2y))}{\partial x}, \quad \text{by result (2),}$$

$$= 2 + z\sin(x^2y) + xz\sin'(x^2y)\frac{\partial(x^2y)}{\partial x}, \quad \text{by result (3)}$$

$$= 2 + z\sin(x^2y) + 2x^2yz\cos(x^2y).$$

Summary

Notation

$g(f)$	Page 1, line - 4
$g \circ f$	Text, page 5
f^{-1}	Page 2, line 16
E^3	Page 3, Definition 1.1
p	Page 3, Definition 1.1
(p_1, p_2, p_3)	Page 3, Definition 1.1
$p + q$	Page 3, line - 3
$(p_1, p_2, p_3) + (q_1, q_2, q_3)$	Page 3, line - 3
ap	Page 3, line - 1
$a(p_1, p_2, p_3)$	Page 3, line - 1
x, y, z or x_1, x_2, x_3	Page 4, Definition 1.2
$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$	Page 4, line 24
$f + g$	Page 4, line - 5
fg	Page 4, line - 5
E^2	Page 5, line 22
$E^1 = R$	Page 5, line 23

Definitions

(i)	Euclidean 3-space E^3	Page 3, Definition 1.1
(ii)	Sum of points of E^3	Page 3, line - 5
(iii)	Scalar product of a point of E^3	Page 3, line - 2
(iv)	Euclidean plane E^2	Page 5, line 22
(v)	Real line $E^1 = R$	Page 5, line 23
(vi)	Natural coordinate functions x, y, z or x_1, x_2, x_3	Page 4, Definition 1.2
(vii)	Pointwise addition and multiplication of functions	Page 4, line - 5
(viii)	Composite of functions $g(f)$	Page 1, line - 4
(ix)	Differentiable function	Page 4, Definition 1.3
(x)	Inverse function f^{-1}	Page 2, line 16

Results

Partial differentiation satisfies the following rules.

(1) $$\frac{\partial(af+bg)}{\partial x_i} = a\frac{\partial f}{\partial x_i} + b\frac{\partial g}{\partial x_i} \qquad (i = 1, 2, 3, \ldots)$$

(2) $$\frac{\partial(fg)}{\partial x_i} = \frac{\partial f}{\partial x_i}g + f\frac{\partial g}{\partial x_i} \qquad (i = 1, 2, 3, \ldots)$$

(3) $$\frac{\partial(h(f))}{\partial x_i} = h'(f)\frac{\partial f}{\partial x_i} \qquad (i = 1, 2, 3, \ldots)$$

where a, b are real numbers, $f, g : E^3 \longrightarrow R$ and $h : R \longrightarrow R$. Text: page 5

Techniques

(i) Evaluation of a function.
(ii) Simplification of a function.
(iii) Partial differentiation, using results (1), (2), (3).

Exercises

Technique (i)

1. Page 5, Exercises 2(b) and 2(c).

Technique (ii)

2. Page 5, Exercise 1(a).
3. Page 6, Exercise 4(a). Just simplify this function.

Technique (iii)

4. Page 5, Exercises 1(c) and 1(d).
5. Page 6, Exercises 3(a) and 3(b).
6. Page 6, Exercise 4(a).

Solutions

1. Page 5, Exercises 2(b) and 2(c).

If p is a point of E^3, then by the pointwise definition of addition and multiplication of functions

$$f(p) = ((x(p))^2\ y(p)) - ((y(p))^2\ z(p)).$$

By Definition 1.2

$$x((3,-1,\tfrac{1}{2})) = 3, \qquad y((3,-1,\tfrac{1}{2})) = -1 \quad \text{and} \quad z((3,-1,\tfrac{1}{2})) = \tfrac{1}{2}.$$

Hence

$$f((3,-1,\tfrac{1}{2})) = (3^2.(-1)) - ((-1)^2.\tfrac{1}{2}) = -9\tfrac{1}{2}.$$

As in the above result

$$f((a, 1, 1-a)) = a^2.1 - 1^2.(1-a) = a^2 + a - 1, \text{ for any real number a.}$$

2. Page 5, Exercise 1(a).

If $f = x^2 y$ then, since f is defined by pointwise multiplication, we have

$$f(p) = x^2(p)\ y(p) = p_1^2\, p_2, \text{ where } p = (p_1, p_2, p_3) \text{ is a point of } E^3.$$

The function $g = y \sin z$ is defined in terms of composition and pointwise multiplication and so

$$g(p) = y(p) \sin (z(p)) = p_2 \sin p_3.$$

Hence

$$\begin{aligned}(fg^2)(p) &= f(p)(g(p))^2 \\ &= p_1^2 p_2\ (p_2 \sin p_3)^2 \\ &= p_1^2 p_2^3 \sin^2 p_3\end{aligned}$$

and

$$fg^2 = x^2y^3 \sin^2 z.$$

In this way we see that pointwise addition and multiplication behave exactly as expected.
If $f = x^2y$ and $g = y \sin z$ then

$$fg^2 = x^2y(y \sin z)^2 = x^2y^3\sin^2 z.$$

3. Page 6, Exercise 4(a).
If $h = x^2 - yz$ then, by the pointwise definitions of addition and multiplication of functions

$$h(\mathbf{p}) = p_1^2 - p_2p_3, \text{ for any point } \mathbf{p} = (p_1, p_2, p_3) \text{ in } E^3.$$

Since $f(\mathbf{p}) = h(g_1(\mathbf{p}), g_2(\mathbf{p}), g_3(\mathbf{p}))$ it follows that

$$f(\mathbf{p}) = (g_1(\mathbf{p}))^2 - (g_2(\mathbf{p}))(g_3(\mathbf{p})).$$

Hence $f = g_1^2 - g_2g_3$.

In this exercise $g_1 = x + y$, $g_2 = y^2$, $g_3 = x + z$ and so

$$\begin{aligned} f &= (x + y)^2 - y^2(x + z) \\ &= x^2 + 2xy + y^2 - xy^2 - y^2z. \end{aligned}$$

4. Page 5, Exercises 1(c) and 1(d).
Now $fg = x^2y^2 \sin z$.
Treating x and y as constants we find that

$$\frac{\partial(fg)}{\partial z} = x^2y^2 \cos z.$$

Treating x and z as constants we find that

$$\frac{\partial^2(fg)}{\partial y \partial z} = 2x^2y \cos z.$$

Now $\sin f = \sin x^2y$.
Treating x as a constant we find that

$$\frac{\partial(\sin f)}{\partial y} = x^2 \cos x^2y.$$

5. Page 6, Exercises 3(a) and 3(b).
If $f = x \sin(xy) + y \cos(xz)$ then, treating y and z as constants,

$$\frac{\partial f}{\partial x} = xy \cos(xy) - yz \sin(xz).$$

To deal with Exercise 3(b) we need to extend result (3) to cover the partial differentiation of the composite of three functions, h from E^3 to R and h_1, h_2 from R to itself. In this exercise it will be more convenient to use the symbol $\circ$ for the composite of functions.

$$\frac{\partial(h_2 \circ (h_1 \circ h))}{\partial x} = (h_2' \circ (h_1 \circ h)) \frac{\partial(h_1 \circ h)}{\partial x}, \qquad \text{by result (3),}$$

$$= (h_2' \circ (h_1 \circ h)).\,(h_1' \circ h)\frac{\partial h}{\partial x}, \qquad \text{by result (3).}$$

Here $h = x^2 + y^2 + z^2$, $h_1 = \exp$, $h_2 = \sin$

and $f = \sin \circ (\exp \circ (x^2 + y^2 + z^2)) = h_2 \circ (h_1 \circ h)$. Hence

$$\frac{\partial f}{\partial x} = (\sin' \circ (\exp \circ (x^2 + y^2 + z^2)).\,\exp' \circ (x^2 + y^2 + z^2).\,\frac{\partial(x^2 + y^2 + z^2)}{\partial x}$$

$$= \cos(e^{(x^2+y^2+z^2)})\; e^{(x^2+y^2+z^2)} 2x$$

$$= 2x\, e^{(x^2+y^2+z^2)} \cos\,(e^{(x^2+y^2+z^2)}).$$

6. Page 6, Exercise 4(a).
From Exercise 3

$$f = x^2 + 2xy + y^2 - xy^2 - y^2 z$$

hence

$$\frac{\partial f}{\partial x} = 2x + 2y - y^2.$$

I.2 TANGENT VECTORS

Introduction

In this section we review *arrows*. They will be called *tangent vectors*. At each point of Euclidean space they form a vector space. We also review *vector fields*. They are functions assigning to each point of Euclidean space a tangent vector based at that point. We generalize the notion of basis and coordinates to deal with vector fields.

READ: Section I.2 (pages 6–10).

Comments

(i) *Page 9: Definition 2.4* Whereas a vector field V assigns *one* tangent vector to each point of Euclidean space, the natural frame field assigns *three*.

(ii) *Page 10: end of Section* Since any vector field V can be expressed as $V = \Sigma v_i U_i$, for suitable real valued functions v_1, v_2 and v_3 the natural frame field does serve as a 'basis'. When we used the term *basis* for a real vector space it implied that every vector could be expressed as a linear combination of the basis vectors. The values of the vector fields U_1, U_2, U_3 are linearly independent at each point. However, not every vector field can be expressed as a linear combination of the U_i's with constant coefficients. That is the coefficients in the sum $\Sigma v_i U_i$ are not necessarily real numbers. Now they are real valued functions and the collection of real valued functions is not even a field since it is not possible to find a multiplicative inverse of a function which is zero at some points even if it is not always zero. These functions form a *ring* and the vector fields are a finite-dimensional *module* over this ring.

Supplementary Comment

(i) *Page 10: lines 3 and 4* These are generalizations of the usual rules for the addition and scalar multiplication of vectors expressed in terms of coordinates.

The first equation is obtained as follows:

$(\Sigma(v_i + w_i)U_i)(p) = \Sigma(v_i + w_i)(p)U_i(p)$, by repeated use of the pointwise formulas for addition and scalar multiplication of vector fields,

$= \Sigma(v_i(p) + w_i(p))U_i(p)$, by the definition of pointwise addition of two functions,

$= \Sigma v_i(p)U_i(p) + \Sigma w_i(p)U_i(p)$, by the usual rules for a vector space,

$= (\Sigma v_i U_i)(p) + (\Sigma w_i U_i)(p)$

$= (\Sigma v_i U_i + \Sigma w_i U_i)(p)$, by the pointwise rules for the "arithmetic" of vector fields.

Since vector fields are defined as functions on E^3, whenever two vector fields agree at each point they must be the same vector field.

Hence

$$\Sigma(v_i + w_i)U_i = \Sigma v_iU_i + w_iU_i.$$

The result

$$\Sigma(fv_i)U_i = f(\Sigma v_iU_i)$$

follows similarly.

Summary

Notation

v_p	Page 6, Definition 2.1
$T_p(E^3)$	Page 7, Definition 2.2
$v_p + w_p$	Page 7, line 9
av_p	Page 7, line 9
V	Page 8, Definition 2.3
$V + W$	Page 8, line -8
fV	Page 8, line -3
$\{U_1, U_2, U_3\}$	Page 9, Definition 2.4
Σv_iU_i	Page 9, line -8

Definitions

(i)	Tangent vector v_p	Page 6, Definition 2.1
(ii)	Vector part of tangent vector	Page 6, Definition 2.1
(iii)	Point of application of tangent vector	Page 6, Definition 2.1
(iv)	Parallel tangent vectors	Page 6, line -5
(v)	Tangent space $T_p(E^3)$	Page 7, Definition 2.2
(vi)	Addition and scalar multiplication of tangent vectors	Page 7, line 9
(vii)	Vector field V	Page 8, Definition 2.3
(viii)	Pointwise principle	Page 8, line -7
(ix)	Addition of vector fields	Page 8, line -10
(x)	Multiplication of a vector field by a real-valued function	Page 8, line -3
(xi)	Natural frame field $\{U_1, U_2, U_3\}$	Page 9, Definition 2.4
(xii)	Euclidean coordinate functions	Page 9, Lemma 2.5
(xiii)	Differentiable vector field	Page 10, line 6

Results

(i)	$T_p(E^3)$ is linearly isomorphic to E^3.	Page 7, line −3
(ii)	Each vector field V can be uniquely expressed as $V = \Sigma v_i U_i$, where v_1, v_2, v_3 are real-valued functions.	Page 9, Lemma 2.5
(iii)	Each tangent vector $(a_1, a_2, a_3)_p$ can be expressed in terms of the natural frame field as $\Sigma a_i U_i(p)$.	Page 9, line −5
(iv)	Addition and multiplication by a function are expressed in terms of coordinates by $\Sigma v_i U_i + \Sigma w_i U_i = \Sigma(v_i + w_i)U_i$ $f(\Sigma v_i U_i) = \Sigma(f v_i)U_i$.	Page 10, lines 3 and 4

Techniques

(i)	Representation of tangent vectors in standard form v_p.	Page 6, Definition 2.1
(ii)	Addition and scalar multiplication of tangent vectors.	Page 7, line 9
(iii)	Evaluation of a vector field at a point.	Page 8, Definition 2.3
(iv)	Addition and multiplication, by real-valued functions, of vector fields.	Page 8, line -8 and Page 9, line 2
(v)	Pictorial representation of tangent vectors and vector fields.	Page 7, Fig. 1.1
(vi)	Representation of tangent vectors and vector fields in terms of the natural frame field and Euclidean coordinates.	Page 9, Lemma 2.5
(vii)	Addition and multiplication, by a real-valued function, of vector fields expressed in terms of Euclidean coordinates.	Page 10, lines 3 and 4

Exercises

Techniques (ii), (vi) and (v)

1. Page 10, Exercise 1.

Techniques (iii) and (vii)

2. Page 10, Exercise 2.

Techniques (vii), (vi), (iv) and (i)

3. Page 10, Exercise 3.

Solutions

1. Page 10, Exercise 1.

(a)

$$\begin{aligned} 3\mathbf{v} - 2\mathbf{w} &= 3(-2, 1, -1) - 2(0, 1, 3) \\ &= (-6, 3, -3) - (0, 2, 6) \\ &= (-6, 1, -9). \end{aligned}$$

By the definition of addition and scalar multiplication of tangent vectors,

$$3\mathbf{v}_{\mathbf{p}} - 2\mathbf{w}_{\mathbf{p}} = (3\mathbf{v} - 2\mathbf{w})_{\mathbf{p}} = (-6, 1, -9)_{\mathbf{p}}.$$

Using the identity $(a_1, a_2, a_3)_{\mathbf{p}} = \Sigma a_i U_i(\mathbf{p})$, we find that

$$3\mathbf{v}_{\mathbf{p}} - 2\mathbf{w}_{\mathbf{p}} = -6U_1(\mathbf{p}) + U_2(\mathbf{p}) - 9U_3(\mathbf{p}).$$

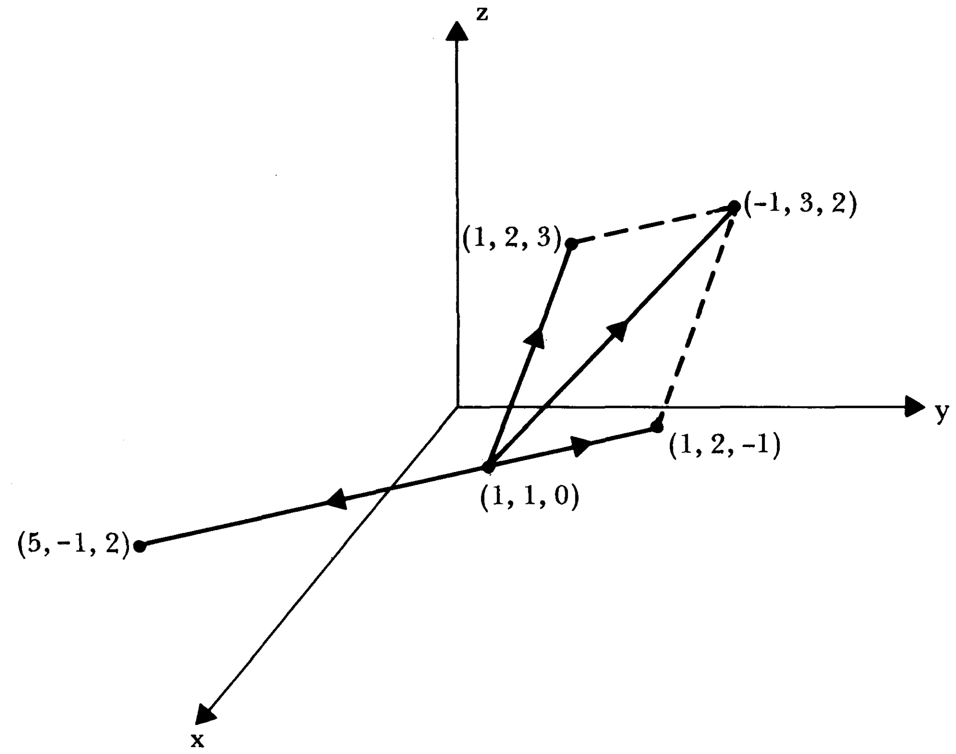
(b) $\mathbf{v}_{\mathbf{p}}$ represents the arrow from the point $\mathbf{p}$ to the point $\mathbf{p} + \mathbf{v}$.
Here $\mathbf{p} = (1, 1, 0)$, $\mathbf{v} = (-2, 1, -1)$ and hence $\mathbf{p} + \mathbf{v} = (-1, 2, -1)$.
Similarly, $\mathbf{w}_{\mathbf{p}}$ is the arrow from $\mathbf{p} = (1, 1, 0)$ to $\mathbf{p} + \mathbf{w} = (1, 2, 3)$.
Now

$$-2\mathbf{v}_{\mathbf{p}} = (-2\mathbf{v})_{\mathbf{p}} = (4, -2, 2)_{\mathbf{p}}$$

and

$$\mathbf{v}_{\mathbf{p}} + \mathbf{w}_{\mathbf{p}} = (\mathbf{v} + \mathbf{w})_{\mathbf{p}} = (-2, 2, 2)_{\mathbf{p}}$$

and hence they are the arrows from (1, 1, 0) to (5, -1, 2) and (-1, 3, 2).



2. Page 10, Exercise 2.

$$\begin{aligned} W - xV &= 2x^2U_2 - U_3 - x(xU_1 + yU_2) \\ &= 2x^2U_2 - U_3 - x^2U_1 - xyU_2 \\ &= -x^2U_1 + (2x^2 - xy)U_2 - U_3, \end{aligned}$$

by result (iv) on the addition and multiplication of vector fields in terms of coordinates.

If $p = (-1, 0, 2)$ then $x(p) = -1$, $y(p) = 0$ and $z(p) = 2$.

By the definition of the natural frame field

$U_1(p) = (1, 0, 0)_p$, $U_2(p) = (0, 1, 0)_p$, $U_3(p) = (0, 0, 1)_p$. Hence by the pointwise definition of addition and multiplication

$$\begin{aligned} (W - xV)((-1, 0, 2)) &= -(-1)^2(1, 0, 0)_p + (2.(-1)^2 - (-1).0)(0, 1, 0)_p - (0, 0, 1)_p \\ &= (-1, 2, -1)_p. \end{aligned}$$

3. Page 10, Exercise 3.

(a) Rearranging and dividing by 7 we obtain

$$V = \frac{2z^2}{7}U_1 - \frac{xy}{7}U_3.$$

(b)
$$\begin{aligned} V(p) &= (p_1, p_3 - p_1, 0)_p \\ &= (x_1(p), x_3(p) - x_1(p), 0)_p, && \text{using the definition of the coordinate functions,} \\ &= (x_1(p), (x_3 - x_1)(p), 0)_p, && \text{by the definition of linear combinations of functions,} \\ &= x_1(p)U_1(p) + (x_3 - x_1)(p)U_2(p), && \text{by the standard identity } (a_1, a_2, a_3)_p = \Sigma a_iU_i(p). \end{aligned}$$

Hence, using the definition of linear combinations of vector fields,

$$V = x_1U_1 + (x_3 - x_1)U_2.$$

Alternatively we could write this as

$$V = xU_1 + (z - x)U_2.$$

(c)
$$\begin{aligned} V &= 2(xU_1 + yU_2) - x(U_1 - y^2U_3) \\ &= 2xU_1 + 2yU_2 - xU_1 + xy^2U_3, && \text{by result (iii) on addition and multiplication in terms of coordinates,} \\ &= (2x - x)U_1 + 2yU_2 + xy^2U_3 \\ &= xU_1 + 2yU_2 + xy^2U_3. \end{aligned}$$

(d) If $p = (p_1, p_2, p_3)$ then $V(p)$ is the vector from (p_1, p_2, p_3) to $(1 + p_1, p_2p_3, p_2)$.

Hence $V(\mathbf{p}) = ((1 + p_1, p_2p_3, p_2) - (p_1, p_2, p_3))_{\mathbf{p}}$

$= (1, p_2p_3 - p_2, p_2 - p_3)_{\mathbf{p}}$.

By arguments similar to those in part (b) we find that

$$V = U_1 + x_2(x_3 - 1)U_2 + (x_2 - x_3)U_3$$

or alternatively

$$V = U_1 + y(z - 1)U_2 + (y - z)U_3.$$

(e) The vector from $\mathbf{p}$ to the origin is $\mathbf{0} - \mathbf{p} = -\mathbf{p}$ so

$$V(\mathbf{p}) = (-\mathbf{p})_{\mathbf{p}} = (-p_1, -p_2, -p_3)_{\mathbf{p}}.$$

Hence $V = -x_1U_1 - x_2U_2 - x_3U_3$ or alternatively

$= -xU_1 - yU_2 - zU_3$.

I.3 DIRECTIONAL DERIVATIVES

Introduction

This section follows on from Sections I.1 and I.2. In Section I.1 we reviewed partial differentiation. We recalled that for any differentiable function f on E^3 and for any point p in E^3

$$\frac{\partial f}{\partial x_1}(p) = \frac{d}{dt}(f(t, p_2, p_3))\Big|_{t=p_1}.$$

Instead of evaluating this derivative at $t = p_1$ we often 'shift' the function so that we can evaluate the derivative at t = 0. That is

$$\frac{\partial f}{\partial x_1}(p) = \frac{d}{dt}(f(p_1 + t, p_2, p_3))\Big|_{t=0}.$$

Using the natural frame field introduced in Section I.2 we can write this as

$$\frac{\partial f}{\partial x_1}(p) = \frac{d}{dt}(f(p + tU_1(p)))\Big|_{t=0}.$$

There are similar formulas for $(\partial f/\partial x_2)(p)$ and $(\partial f/\partial x_3)(p)$, so to find $(\partial f/\partial x_i)(p)$ we find the initial rate of change of the function f in the direction $U_i(p)$. In *Unit MST 282 7, Work and Energy I*, Section 7.1.2 we met a generalization of this technique. If **n** was a unit direction vector we defined the directional derivative, $(\partial f/\partial \mathbf{n})(p)$, to be the initial rate of change of f in the direction **n** and proved that

$$\frac{\partial f}{\partial \mathbf{n}} = \mathbf{n} \cdot \nabla f(p) = \sum_{i=1}^{3} n_i \frac{\partial f}{\partial x_i}(p).$$

In this section we generalize the above definition and result to deal with any tangent vector v_p and then any vector field V. We show that these directional derivatives satisfy properties similar to those of ordinary derivatives.

READ: Section I.3 (pages 11-14) omitting the proof of Lemma 3.2.

Comments

(i) *Page 12: Lemma 3.2* This is a generalization of the result mentioned in the introduction. The directional derivative is

$$v_p[f] = v \cdot \nabla f(p) = \sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_i}(p).$$

It is proved in exactly the same way using a version of the 'chain rule'. This will be dealt with in Section I.7 and until then you should accept this result without proof.

(ii) *Page 12: Theorem 3.3.(3)* The symbol $\cdot$ in this theorem and in Corollary 3.4 signifies the ordinary product of two numbers. The inner product of two vectors will not be introduced until Chapter II.

(iii) *Page 12: line 15 to line -3* The real-valued function V[f] is defined by

$$(V[f])(p) = V(p)[f].$$

This equation is used in the proof of Corollary 3.4.

For computational purposes the identities $U_i[f] = \partial f/\partial x_i$ $(i = 1, 2, 3)$ will be very important. We can obtain these results from Lemma 3.2. For instance

$$U_1(p)\,[f] = (1, 0, 0)_p[f] = 1.\frac{\partial f}{\partial x_1}(p) + 0.\frac{\partial f}{\partial x_2}(p) + 0.\frac{\partial f}{\partial x_3}(p) = \frac{\partial f}{\partial x_1}(p).$$

(iv) *Page 14; line 14 to line 21* This is a very important remark. For vector fields V and W we consider combinations of the form $fV + gW$, where f and g are real-valued functions. For functions f and g we consider combinations of the form $af + bg$, where a and b are real numbers.

In the example

$$(xU_1 - y^2U_3)[x^2y + z^3] = xU_1[x^2y + z^3] - y^2U_3[x^2y + z^3],$$

by Corollary 3.4.(1),

$$= xU_1[x^2y] + xU_1[z^3] - y^2U_3[x^2y] - y^2U_3[z^3],$$

by Corollary 3.4.(2).

Supplementary Comment

(i) *Page 12: Theorem 3.3.(1) and (2)*

(1) If $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ then

$$(av_p + bw_p)[f] = (av + bw)_p[f],$$

by the definition of addition and scalar multiplication of tangent vectors,

$$= \Sigma(av_i + bw_i)\frac{\partial f}{\partial x_i}(p),$$

by Lemma I.3.2,

$$= a\left(\Sigma v_i\frac{\partial f}{\partial x_i}(p)\right) + b\left(\Sigma w_i\frac{\partial f}{\partial x_i}(p)\right)$$

$$= av_p[f] + bw_p[f].$$

(2) $$v_p[af + bg] = \Sigma v_i\left(\frac{\partial(af + bg)}{\partial x_i}\right)(p),$$ by Lemma I.3.2,

$$= \Sigma v_i\left(a\frac{\partial f}{\partial x_i}(p) + b\frac{\partial g}{\partial x_i}(p)\right)$$

$$= a\left(\Sigma v_i\frac{\partial f}{\partial x_i}(p)\right) + b\left(\Sigma v_i\frac{\partial g}{\partial x_i}(p)\right).$$

Summary

Notation

$v_p[f]$	Page 11, Definition 3.1
$V[f]$	Page 13, line 15

Definitions

(i)	Derivative of a function with respect to a tangent vector $v_p[f]$.	Page 11, Definition 3.1
(ii)	Derivative of a function with respect to a vector field $V[f]$.	Page 13, line 15

Results

(i)	If $v_p = (v_1, v_2, v_3)_p$ then $v_p[f] = \Sigma v_i \frac{\partial f}{\partial x_i}(p)$.	Page 12, Lemma 3.2
(ii)	Directional derivatives, with respect to tangent vectors and vector fields, satisfy linear and Leibnizian properties:	
	(a) $(av_p + bw_p)[f] = av_p[f] + bw_p[f]$.	
	(b) $v_p[af + bg] = av_p[f] + bv_p[g]$.	
	(c) $v_p[fg] = v_p[f]g(p) + f(p)v_p[g]$.	Page 12, Theorem 3.3
	(a) $(fV + gW)[h] = fV[h] + gW[h]$.	
	(b) $V[af + bg] = aV[f] + bV[g]$.	
	(c) $V[fg] = V[f].g + fV[g]$.	Page 13, Corollary 3.4.
(iii)	Directional derivatives with respect to components of the natural frame field are partial derivatives $U_i[f] = \frac{\partial f}{\partial x_i}$ $(i = 1, 2, 3)$.	Page 13, line -10

Techniques

Evaluation of directional derivatives with respect to tangent vectors or vector fields using:

(i)	the definition, for tangent vectors	Page 11, Definition 3.1
(ii)	the definition, for vector fields	Text, page 16
(iii)	coordinates and partial derivatives	Page 12, Lemma 3.2
(iv)	the linear and Leibnizian properties and the relationship between the natural frame fields and partial derivatives.	Page 12, Theorem 3.3, Page 13, Corollary 3.4 and Page 13, line -10.

Exercises

Technique (i)

1. Page 14, Exercise 1(c).

Technique (ii)

2. Determine $V[f]$ where $f = e^x \cos y$ and $V = 2U_1 - U_2 + 3U_3$.

Technique (iii)

3. Page 15, Exercise 2(c).

Technique (iv)

4. Page 15, Exercise 3(e). First prove that $V[f^2] = 2f\,V[f]$.

Theory Exercises

5. Page 15, Exercise 4.

6. Page 15, Exercise 5. (HINT: Use Exercise 4.)

Solutions

1. Page 14, Exercise 1(c).

By Definition I.3.1,

$$v_p[f] = \frac{d}{dt}(f(p + tv))\Big|_{t=0}.$$

Here

$$p + tv = (2, 0, -1) + t(2, -1, 3) = (2 + 2t, -t, -1 + 3t)$$

and

$$f(p + tv) = (e^x\cos y)\,(2 + 2t, -t, -1 + 3t)$$

$$= e^{2+2t}\cos(-t),$$ by the pointwise definition of addition, multiplication and composition of functions and the definition of the functions x and y,

$$= e^2e^{2t}\cos t.$$

Hence

$$v_p[f] = e^2\frac{d}{dt}(e^{2t}\cos t)\Big|_{t=0}$$

$$= e^2(2e^{2t}\cos t - e^{2t}\sin t)\Big|_{t=0} = 2e^2.$$

2. The value of the function $V[f]$ at a point p is given by the formula $(V[f])(p) = V(p)[f]$.

Now $V(p) = 2U_1(p) - U_2(p) + 3U_3(p) = (2, -1, 3)_p$.

Hence if $p = (p_1, p_2, p_3)$ then

$$V(p)[f] = \frac{d}{dt}(f((p_1, p_2, p_3) + t(2, -1, 3)))\Big|_{t=0}$$

$$= \frac{d}{dt}(f(p_1 + 2t, p_2 - t, p_3 + 3t))\Big|_{t=0}$$

$$= \frac{d}{dt}(e^{p_1+2t}\cos(p_2 - t))\Big|_{t=0}$$

$$= (2e^{p_1+2t}\cos(p_2 - t) + e^{p_1+2t}\sin(p_2 - t))\Big|_{t=0}$$

$$= e^{p_1}(2\cos p_2 + \sin p_2).$$

So $V[f](p) = e^{p_1}(2\cos p_2 + \sin p_2)$ and hence

$$V[f] = e^x(2\cos y + \sin y).$$

3. Page 15, Exercise 2(c).
By Lemma I.3.2

$$v_p[f] = \Sigma v_i \frac{\partial f}{\partial x_i}(p),$$

though in this solution we replace x_1, x_2, x_3 by x, y, z. Since $f = e^x \cos y$ it follows that

$$\frac{\partial f}{\partial x} = e^x \cos y, \qquad \frac{\partial f}{\partial y} = -e^x \sin y \quad \text{and} \quad \frac{\partial f}{\partial z} = 0.$$

Since $p = (2, 0, -1)$

$$\frac{\partial f}{\partial x}(p) = e^2, \qquad \frac{\partial f}{\partial y}(p) = 0 \quad \text{and} \quad \frac{\partial f}{\partial z}(p) = 0.$$

Hence, since $v = (2, -1, 3)$

$$v_p[f] = 2.e^2 + (-1).0 + 3.0 = 2e^2.$$

4. Page 15, Exercise 3(e).
By the Leibnizian property, Corollary I.3.4(3),

$$V[f^2] = V[f.f] = V[f].f + f\,V[f] = 2f\,V[f].$$

Hence by the linearity property, Corollary I.3.4(2),

$$V[f^2 + g^2] = 2(f\,V[f] + g\,V[g]).$$

Now

$$\begin{aligned} V[f] &= (y^2U_1 - xU_3)[xy] = y^2U_1[xy] - xU_3[xy] \\ &= y^2\frac{\partial(xy)}{\partial x} - x\frac{\partial(xy)}{\partial z}, \qquad \text{since } U_i[f] = \frac{\partial f}{\partial x_i}, \\ &= y^2.y - x.0 = y^3. \end{aligned}$$

Similarly,

$$\begin{aligned} V[g] &= (y^2U_1 - xU_3)[z^3] \\ &= -3xz^2 \end{aligned}$$

and hence

$$\begin{aligned} V[f^2 + g^2] &= 2(xy.(y^3) + z^3.(-3xz^2)) \\ &= 2x(y^4 - 3z^5). \end{aligned}$$

5. Page 15, Exercise 4.
Following the hint we evaluate $V[x_j]$, where $V = \Sigma v_i U_i$.

For fixed j we have

$$V[x_j] = \Sigma v_i U_i[x_j] = \Sigma v_i \frac{\partial x_j}{\partial x_i}.$$

Now $\frac{\partial x_j}{\partial x_i} = 1$ if $i = j$ and is zero otherwise. Hence

$$V[x_j] = v_j \qquad \text{for all } j,$$

and

$$V = \Sigma v_i U_i = \Sigma V[x_i]\, U_i.$$

6. Page 15, Exercise 5.
Since $V[f] = W[f]$ for every function f on E^3 it follows that

$$V[x_i] = W[x_i] \qquad (i = 1, 2, 3).$$

Hence by the result of Exercise 4

$$V = \Sigma V[x_i]\, U_i = \Sigma W[x_i]\, U_i = W.$$

I.4 CURVES IN E^3

Introduction

This section follows on from Sections I.2 and I.3. We review two ways of defining curves, the definition of the velocity of a curve, and show how the rate of change of a function along a curve is related to the directional derivative with respect to the velocity vector.

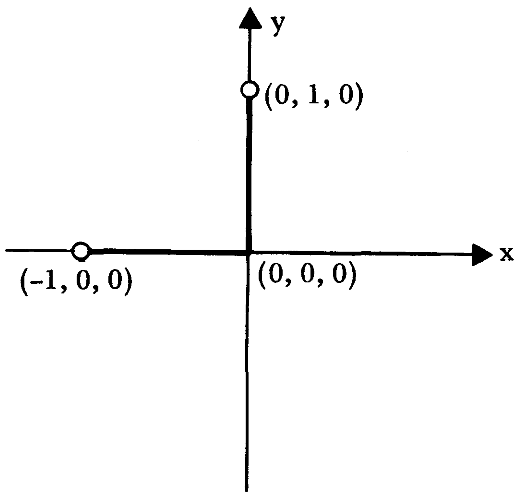
READ: Section I. 4 (pages 15-21) omitting the proof of Lemma 4.6.

Comments

(i) *Section I.4* From Definition 4.1 you might think that curves always have routes that appear in some sense smooth. However, curves can be degenerate in several different ways. For instance, the simplest curve is the constant curve $\alpha : t \longmapsto 0$, which has as its route a single point. The curve $\alpha : t \longmapsto (\sin t, 0, 0)$ has as its route a closed interval which it traverses infinitely often. It is even possible to have a curve that turns sharp corners. For instance, the curve

$$\alpha : t \longmapsto \begin{cases} (-e^{-1/t^2}, 0, 0) & \text{for } t < 0 \\ (0, 0, 0) & \text{for } t = 0 \\ (0, e^{-1/t^2}, 0) & \text{for } t > 0 \end{cases}$$

has the following route



This function was chosen to ensure that the Euclidean coordinate functions are 'infinitely' differentiable. In order for the curve to stand still, reverse or turn a sharp corner, as in the above examples, it is necessary for the velocity vector to take the value zero. Hence, if we restrict attention to regular curves all these pathologies can be avoided. The definition of a 'Curve' ensures that we can always find a regular parametrization of each component. We use the word "closed" to describe components of a Curve that are like a distorted circle and so not surprisingly they have periodic parametrizations. Any component that is not like a distorted circle is like a distorted copy of the real line and so not surprisingly has a one-to-one parametrization. The circle and a branch of the hyperbola, described on page 21, are typical of closed and not closed Curves.

(ii) *Page 15: line 9 of the section* Here the Euclidean coordinate functions are real-valued functions on some interval I. When we described a vector field in terms of its Euclidean coordinate functions they were real-valued functions on E^3.

Supplementary Comments

(i) *Page 16: Example 4.2(3)*

$$\alpha(t) = (2\cos^2 t, \sin 2t, 2\sin t) = (2\cos^2 t, 2\sin t \cos t, 2\sin t).$$

Since

$$(2\cos^2 t)^2 + (2\sin t\ \cos t)^2 + (2\sin t)^2$$
$$= 4\cos^2 t(\cos^2 t + \sin^2 t) + 4\sin^2 t = 4(\cos^2 t + \sin^2 t) = 4$$

it follows that $\alpha(t)$ belongs to S.

Also

$\alpha(t) - (1,0,0) = (2\cos^2 t - 1, \sin 2t, 2\sin t) = (\cos 2t, \sin 2t, 2\sin t)$ and since $\cos^2 2t + \sin^2 2t = 1$ it follows that $\alpha(t)$ belongs to C.

As t tends to 0 the point $\alpha(t)$ tends to (2,0,0), and as t tends to $\pi/2$ the point $\alpha(t)$ tends to (0,0,2), and hence α follows the route sliced from C by the sphere S.

(ii) *Page 18: line 6* Since $\alpha_i(t) = p_i + q_i$ $(i = 1, 2, 3)$ it follows that $d\alpha_i/dt(t) = q_i$ $(i = 1, 2, 3)$ and hence $\alpha'(t) = (q_1, q_2, q_3)_{\alpha(t)}$.

(iii) *Page 19: Lemma 4.5* By the definition of the composite of two functions,

$$(\alpha(h))(s) = \alpha(h(s)) \text{ and } (\alpha_i(h))(s) = \alpha_i(h(s)).$$

Hence, since $\beta(s) = \alpha(h(s)) = (\alpha(h))(s)$,

$$\beta'(s) = (\alpha(h))'(s).$$

Since $\alpha(h(s)) = (\alpha_1(h(s)), \alpha_2(h(s)), \alpha_3(h(s)))$,

$$(\alpha(h))(s) = ((\alpha_1(h))(s), (\alpha_2(h))(s), (\alpha_3(h))(s))$$

and by Definition 4.3

$$(\alpha(h))'(s) = ((\alpha_1(h))'(s), (\alpha_2(h))'(s), (\alpha_3(h))'(s)).$$

The result now follows as on page 19.

Summary

Notation

$\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$	Page 15, line 7
$\alpha = (\alpha_1, \alpha_2, \alpha_3)$	Page 15, line 10
α	Page 15, Definition 4.1
$\alpha'(t)$	Page 17, Definition 4.3
C: f = a	Page 20, line -5.

Definitions

(i)	Euclidean coordinate functions of a curve $\alpha = (\alpha_1, \alpha_2, \alpha_3)$	Page 15, line 9
(ii)	Curve α	Page 15, Definition 4.1
(iii)	Velocity vector of a curve $\alpha'(t)$	Page 17, Definition 4.3
(iv)	Reparametrization of a curve	Page 18, Definition 4.4
(v)	Periodic curve	Page 20, line 13
(vi)	Period of a curve	Page 20, line 15
(vii)	Regular curve	Page 20, line -10
(viii)	'Curves' on the plane defined implicitly C: $f = a$	Page 20, line -5
(ix)	Closed curve on the plane	Page 21, line 5 and Text: page 22

Examples

(i)	Straight line. $\alpha(t) = p + tq$	Page 15, Example 4.2.(1)
(ii)	Helix. $\alpha(t) = (a \cos t, a \sin t, bt)$	Page 15, Example 4.2.(1)

Results

(i)	If β is the reparametrization of α by h, then $\beta'(s) = \left(\frac{dh}{ds}\right)(s).\alpha'(h(s)).$	Page 19, Lemma 4.5
(ii)	If α is a curve in E^3 and f is a differentiable function on E^3, then $\alpha'(t)[f] = \frac{d(f(\alpha))}{dt}(t).$	Page 19, Lemma 4.6

Techniques

(i)	Determination of a parametric representation of a straight line.	Page 15, Example 4.2.(1)
(ii)	Determination of the velocity vectors of a curve.	Page 17, Definition 4.3
(iii)	Determination of the effect of reparametrization on velocity vectors.	Page 19, Lemma 4.5
(iv)	Calculation of the rate of change of a function along a curve by means of the directional derivative with respect to the velocity vector.	Page 19, Lemma 4.6
(v)	Parametrization of implicitly defined 'Curves' in E^2.	Page 21.

Exercises

Technique (i)

1. Page 21, Exercise 5.

Technique (ii)

2. Page 21, Exercise 1.

Technique (iii)

3. Page 21, Exercise 3. Evaluate $\alpha'(\pi/4)$, $\beta'(1/\sqrt{2})$ and $(dh/ds)(1/\sqrt{2})$. Verify Lemma I.4.5 in this case.

   (Note: $\sin^{-1}$ was called arc sin in M231 and $(\sin^{-1})'(s) = 1/(1 - s^2)^{\frac{1}{2}}$.)

Technique (iv)

4. Page 21, Exercise 7.

Technique (v)

5. Page 22, Exercise 10.

Solutions

1. Page 21, Exercise 5.

Suppose the line is given by $\alpha(t) = p + tq$. If it passes through the points p_1 and p_2 we can assume $\alpha(0) = p_1$ and $\alpha(1) = p_2$. Then $p = p_1$ and $q = p_2 - p_1$ and hence $\alpha(t) = p_1 + t(p_2 - p_1)$. If $p_1 = (1, -3, -1)$ and $p_2 = (6, 2, 1)$ the curve is

$$\begin{aligned}\alpha(t) &= (1, -3, -1) + t((6, 2, 1) - (1, -3, -1))\\ &= (1, -3, -1) + t(5, 5, 2)\\ &= (1 + 5t, -3 + 5t, -1 + 2t).\end{aligned}$$

Similarly the line which passes through $(-1, 1, 0)$ and $(-5, -1, -1)$ is given by

$$\beta(s) = (-1 - 4s, 1 - 2s, -s).$$

If the lines meet, there are real numbers t and s such that

$$\alpha(t) = \beta(s).$$

i.e.

$$1 + 5t = -1 - 4s$$

$$-3 + 5t = 1 - 2s$$

$$-1 + 2t = -s.$$

The first two equations can be solved uniquely giving $t = 2$, $s = -3$, which is also a solution of the third equation, so the lines meet at $\alpha(2) = \beta(-3) = (11, 7, 3)$.

2. Page 21, Exercise 1.

$\alpha(t) = (2\cos^2 t,\ \sin 2t,\ 2\sin t)$.

Hence by Definition I.4.3,

$$\alpha'(t) = \left(\frac{d}{dt}(2\cos^2 t),\ \frac{d}{dt}(\sin 2t),\ \frac{d}{dt}(2\sin t)\right)_{\alpha(t)}$$

$$= (-4\cos t\sin t,\ 2\cos 2t,\ 2\cos t)_{(2\cos^2 t,\ \sin 2t,\ 2\sin t)}$$

and

$$\alpha'\left(\frac{\pi}{4}\right) = (-2, 0, \sqrt{2})_{(1, 1, \sqrt{2})}.$$

3. Page 21, Exercise 3.

By Definition I.4.4 the reparametrized curve is given by

$$\beta(s) = \alpha(h(s)) = \alpha(\sin^{-1} s).$$

Now $\alpha(t) = (2\cos^2 t,\ \sin 2t,\ 2\sin t)$ and to make progress we express $\alpha(t)$ as a function of $\sin t$.

We find that $\alpha(t) = (2(1 - \sin^2 t),\ 2\sin t\,(1 - \sin^2 t)^{\frac{1}{2}}, 2\sin t)$, using the identities $\cos^2 t + \sin^2 t = 1$ and $\sin 2t = 2\sin t\cos t$.

Hence $\beta(s) = \alpha(\sin^{-1}(s))$

$$= (2(1 - s^2),\ 2s(1 - s^2)^{\frac{1}{2}},\ 2s).$$

The function $\sin^{-1}$ has domain $[-1, 1]$ and so is well defined for $0 < s < 1$ and takes values in the interval $0 < s < \pi/2$.

Now

$$\alpha'(t) = (-4\cos t\sin t,\ 2\cos 2t,\ 2\cos t)$$

and

$$\alpha'\left(\frac{\pi}{4}\right) = \left(-4.\frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}},\ 0,\ 2.\frac{1}{\sqrt{2}}\right) = (-2, 0, \sqrt{2}).$$

Also

$$\beta'(s) = \left(-4s,\ 2(1 - s^2)^{\frac{1}{2}} - \frac{2s^2}{(1 - s^2)^{\frac{1}{2}}},\ 2\right)$$

and

$$\beta'\left(\frac{1}{\sqrt{2}}\right) = (-2\sqrt{2}, 0, 2).$$

Finally

$$\frac{dh}{ds}(s) = \frac{1}{(1 - s^2)^{\frac{1}{2}}} \quad\text{and}\quad \frac{dh}{ds}\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}.$$

Hence

$$\beta'\left(\frac{1}{\sqrt{2}}\right) = \frac{dh}{ds}\left(\frac{1}{\sqrt{2}}\right).\alpha'\left(\frac{\pi}{4}\right) \text{ and, since } h\left(\frac{1}{\sqrt{2}}\right) = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4},$$

this agrees with Lemma I.4.5.

4. Page 21, Exercise 7.

Let $\alpha(t) = (t,\ 1 + t^2,\ t)$,

$\beta(t) = (\sin t,\ \cos t,\ t)$,

$\gamma(t) = (\sinh t,\ \cosh t,\ t)$.

Then $\alpha'(0) = \beta'(0) = \gamma'(0) = (1, 0, 1)_{(0, 1, 0)}$. The curves have the same initial velocity v_p.

By Theorem I.4.6

$$v_p[f] = \frac{d}{dt}(f(\alpha))\Big|_{t=0} = \frac{d}{dt}(f(\beta))\Big|_{t=0} = \frac{d}{dt}(f(\gamma))\Big|_{t=0}$$

and since $f = x^2 - y^2 + z^2$ we have

$$\frac{d}{dt}(f(\alpha))\Big|_{t=0} = \frac{d}{dt}(t^2 - (1 + t^2)^2 + t^2)\Big|_{t=0} = \frac{d}{dt}(-1 - t^4)\Big|_{t=0} = 0,$$

$$\frac{d}{dt}(f(\beta))\Big|_{t=0} = \frac{d}{dt}(\sin^2 t - \cos^2 t + t^2)\Big|_{t=0} = 0,$$

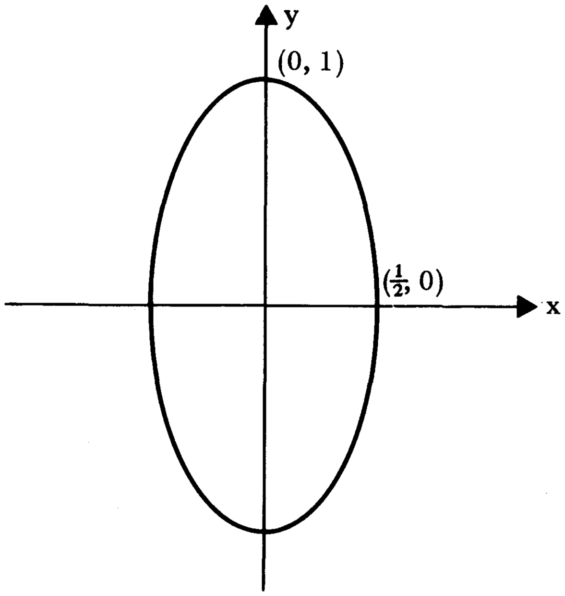
$$\frac{d}{dt}(f(\gamma))\Big|_{t=0} = \frac{d}{dt}(\sinh^2 t - \cosh^2 t + t^2)\Big|_{t=0} = 0.$$

This is the answer we expect since by Lemma I.3.2

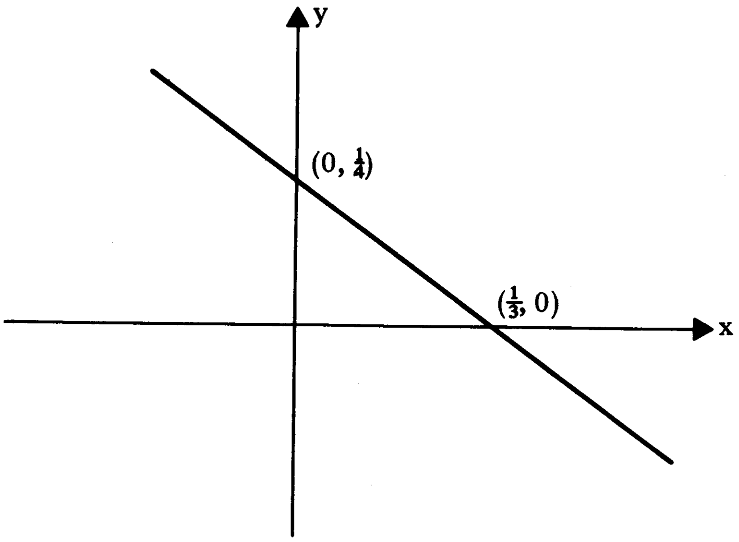
$$\begin{aligned}(1, 0, 1)_{(0, 1, 0)}[f] &= \left(1.\frac{\partial f}{\partial x} + 0.\frac{\partial f}{\partial y} + 1.\frac{\partial f}{\partial z}\right)((0, 1, 0)) \\ &= 2x((0, 1, 0)) = 0.\end{aligned}$$

5. Page 22, Exercise 10.

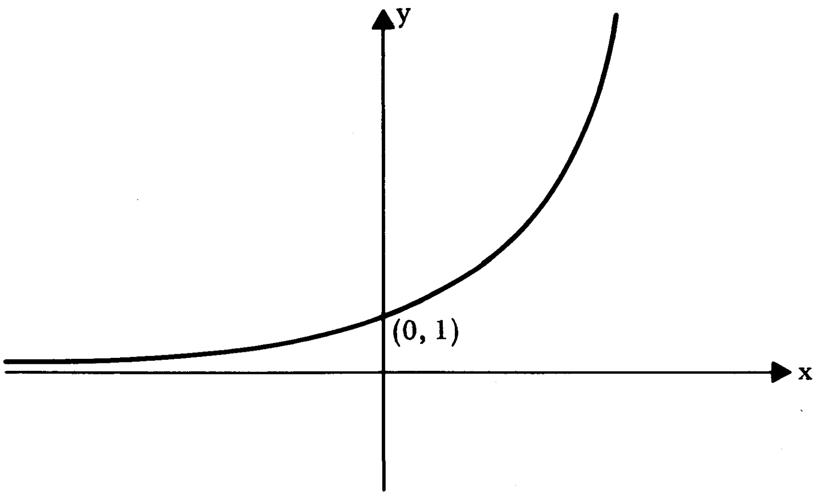
(a) $\alpha : t \longmapsto (\frac{1}{2}\cos t, \sin t)$



(b) $\alpha : t \longmapsto \left(t, \frac{1-3t}{4}\right)$

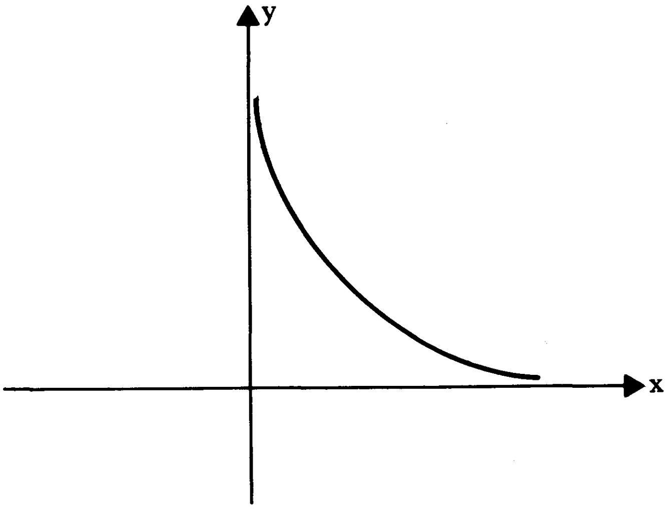


(c) $\alpha : t \longmapsto (t, e^t)$



(d) $\alpha : t \longmapsto (t, (1 - t^{\frac{2}{3}})^{\frac{3}{2}})$

$0 < t < 1$



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I.5 1 -FORMS

Introduction

This section follows on from Section I.3, Directional Derivatives.

In this section we introduce 1-forms. A 1-form is an extension of the idea of linear functional. A linear functional is a linear transformation between a vector space V and the underlying vector space of reals R. Now we have a collection of vector spaces, the tangent spaces $T_p(E^3)$, and we require our 1-forms to map any tangent vector to a real number in such a way that the restriction to any one tangent space is a linear functional.

READ: Section I. 5 (pages 22-25).

Comments

(i) *Page 23: Definition 5.2* If V is a vector field, then the definition of the differential implies that $df(V) = V[f]$. This arises from the pointwise definition in the following way. For any point p in E^3

$$(df(V))(p) = df(V(p)) = V(p)[f] = (V[f])(p)$$

and so $df(V) = V[f]$.

(ii) *Page 23: Example 5.3.(1)* The symbol δ_{ij} is a very useful tool and identities such as

$$\sum_j v_j\delta_{ij} = v_1\delta_{i1} + v_2\delta_{i2} + v_3\delta_{i3} = v_i \qquad (i = 1, 2, 3)$$

occur frequently. To check this identity we look at an example, say i = 2. Then

$$\sum_j v_j\delta_{2j} = v_1\delta_{21} + v_2\delta_{22} + v_3\delta_{23}$$

$$= v_1.0 + v_2.1 + v_3.0 = v_2.$$

(iii) *Page 23: line -1* We use the same symbol for a real number and for the constant function taking that value. Hence we can write $dx_i(U_j) = \delta_{ij}$. We obtain this result from first principles as follows

$$dx_i(U_j) = U_j[x_i] = \frac{\partial x_i}{\partial x_j} = \delta_{ij}.$$

This is the basic result to remember. It can be used to obtain the results of both Example (1) and (2) as follows:

(1) $$dx_i(v_p) = dx_i\left(\sum_j v_jU_j(p)\right) = \sum_j v_j dx_i(U_j(p))$$

$$= \sum_j v_j\delta_{ij} = v_i.$$

(2) $$\psi(v_p) = \left(\sum_i f_i dx_i\right)\left(\sum_j v_jU_j(p)\right)$$

$$= \sum_{i,j} f_i(p)v_j dx_i(U_j(p)) = \sum_{i,j} f_i(p)v_j\delta_{ij}$$

$$= \sum_i f_i(p)v_i.$$

(iv) *Page 25: Lemma 5.7* This follows from the rule for the partial differentiation of a composite given in the Additional Comments to Section I.1.

Supplementary Comments

(i) *Page 23: lines 5 to 10* Two functions are equal if they take the same value at each point p of E^3.

Now

$$\begin{aligned}(\phi(fV+gW))(p) &= \phi((fV+gW)(p)), && \text{by the definition of the evaluation of a 1-form on a vector field,}\\ &= \phi(f(p)V(p)+g(p)W(p)), && \text{by the pointwise definition of operations on vector fields}\\ &= f(p)\phi(V(p))+g(p)\phi(W(p)), && \text{by the definition of a 1-form}\\ &= f(p)(\phi(V))(p)+g(p)(\phi(W))(p)\\ &= (f\phi(V)+g\phi(W))(p).\end{aligned}$$

Hence

$$\phi(fV+gW) = f\,\phi(V) + g\,\phi(W).$$

Similarly, since

$$\begin{aligned}((f\phi+g\psi)(V))(p) &= (f\phi+g\psi)(V(p))\\ &= f(p)\phi(V(p))+g(p)\psi(V(p))\\ &= f(p)(\phi(V))(p)+g(p)(\psi(V))(p)\\ &= (f\phi(V)+g\psi(V))(p),\end{aligned}$$

it follows that $(f\phi+g\psi)(V) = f\phi(V)+g\psi(V)$.

(ii) *Page 23: Definition 5.2* The function df is a 1-form since

$$\begin{aligned}df(av_p+bw_p) &= (av_p+bw_p)[f]\\ &= av_p[f]+bw_p[f], && \text{by Theorem I.3.3.(1),}\\ &= a\,df(v_p)+b\,df(w_p).\end{aligned}$$

(iii) *Page 24: the proof of Lemma 5.4* Since the natural frame field forms a basis, when restricted to each vector space, any 1-form that is zero on all three of these vector fields must be the zero 1-form. For j = 1, 2, 3,

$$\begin{aligned}\left(\phi - \sum_i \phi(U_i)dx_i\right)(U_j) &= \phi(U_j) - \sum_i \phi(U_i)dx_i(U_j)\\ &= \phi(U_j) - \sum_i \phi(U_i)\delta_{ij} = \phi(U_j) - \phi(U_j) = 0.\end{aligned}$$

Hence $\phi - \sum_i \phi(U_i)dx_i = 0$ and $\phi = \sum_i \phi(U_i)dx_i$.

(iv) *Page 24: the proof of Corollary 5.5* This can be proved directly using Lemma I.5.4.

$$\begin{aligned} df &= \sum_i df(U_i)dx_i, && \text{by Lemma I.5.4,} \\ &= \sum_i U_i[f]\,dx_i, && \text{by Definition I.5.2,} \\ &= \sum_i \frac{\partial f}{\partial x_i} dx_i. \end{aligned}$$

(v) *Page 25: the worked example* How do we obtain the first term 2xy dx? Now $f = x^2y + \ldots$ and hence $df = d(x^2y) + \ldots = d(x^2)y + x^2dy + \ldots$, by Lemma I.5.6. To evaluate $d(x^2)$ rigorously we need to introduce the squaring function $S : t \longmapsto t^2$ for which $S' : t \longmapsto 2t$.

Then $d(x^2) = d(S(x)) = S'(x)dx$, by Lemma I.5.7,

$$= 2x\,dx.$$

Hence $df = 2xy\,dx + x^2dy + \ldots$

Finally $v_p[f] = df(v_p)$ is evaluated using the result of Example 5.3.(2).

Summary

Notation

ϕ	Page 22, Definition 5.1
$\phi(v_p)$	Page 22, Definition 5.1
ϕ_p	Page 22, line -11
$\phi + \psi$	Page 22, line -7
$f\phi$	Page 22, line -4
$\phi(V)$	Page 22, line -2
df	Page 23, Definition 5.2
$df(v_p)$	Page 23, Definition 5.2
$df(V)$	Text: page 30
dx_i	Page 23, Example 5.3.(1)
δ_{ij}	Page 23, line -12

Definitions

(i)	1-form ϕ	Page 22, Definition 5.2
(ii)	Addition of 1-forms	Page 22, line -7
(iii)	Multiplication of a 1-form by a real-valued function	Page 22, line -4
(iv)	The effect of a 1-form on a vector field	Page 22, line -2
(v)	The differential of a real valued function df	Page 22, Definition 5.2
(vi)	Kronecker delta δ_{ij}	Page 23, line -12

Results

(i)	$\phi(V)$ is linear in both the 1-forms ϕ and vector fields V.	Text: page 31
(ii)	$dx_i(v_p) = v_i$.	Page 23, Example 5.3.(1)
(iii)	If $\phi = \Sigma f_i dx_i$, then $\phi(v_p) = \Sigma f_i(p) v_i$.	Page 23, Example 5.3.(2)
(iv)	$dx_i(U_j) = \delta_{ij}$.	Page 23, line -1
(v)	$\phi = \Sigma \phi(U_i) dx_i$.	Page 24, Lemma 5.4
(vi)	$df = \Sigma \frac{\partial f}{\partial x_i} dx_i$.	Page 24, Corollary 5.5
(vii)	$d(f + g) = df + dg$.	Page 24, line -9
(viii)	$d(fg) = gdf + fdg$.	Page 24, Lemma 5.6
(ix)	If $f : E^3 \longrightarrow R$ and $h : R \longrightarrow R$ then $d(h(f)) = h'(f)df$.	Page 25, Lemma 5.7

Techniques

(i)	Recognition of a 1-form, by comparison with the formula $\phi(av_p + bw_p) = a\phi(v_p) + b\phi(w_p)$.	Page 22, Definition 5.1
(ii)	Expressing a 1-form in standard form using the formula $\phi = \Sigma \phi(U_i) dx_i$.	Page 24, Lemma 5.4
(iii)	Evaluation of the effect of a 1-form on a tangent vector using the formula $\phi(v_p) = \sum_i f_i(p) v_i$ where $\phi = \sum_i f_i dx_i$.	Page 23, Example 5.3.(2)
(iv)	Evaluation of the effect of a 1-form on a vector field using the identity $dx_i(U_j) = \delta_{ij}$ and linearity.	Text: page 30
(v)	Calculation of differentials using the 'chain rule' $df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3$.	Page 24, Corollary 5.5
(vi)	Calculation of differentials using the linear, Leibnizian and composite properties	
	$d(f + g) = df + dg$,	Page 24, line -9
	$d(fg) = g\,df + f\,dg$,	Page 24, Lemma 5.6
	$d(h(f)) = h'(f)df$.	Page 25, Lemma 5.7
(vii)	Evaluation of directional derivatives using differentials $v_p[f] = df(v_p)$.	Page 23, Definition 5.2

Exercises

Techniques (i) and (ii)

1. Page 26, Exercise 7.

Technique (iii)

2. Page 25, Exercise 1(c).

Technique (iv)

3. Page 25, Exercise 3. Evaluate ϕ on W only.

Technique (v)

4. Page 26, Exercise 5(a).

Technique (vi)

5. Page 26, Exercise 5(b). $\left(\text{Note: } (\tan^{-1})'(t) = \frac{1}{1+t^2}.\right)$

Technique (vii)

6. Page 26, Exercise 6(b).

Solutions

1. Page 26, Exercise 7.

To be a 1-form, ϕ needs to be linear on each tangent space $T_p(E^3)$. That is, the mapping

$(v_1, v_2, v_3) \longmapsto \phi((v_1, v_2, v_3)_p)$ must be linear for each p in E^3.

(a) The mapping is

$$(v_1, v_2, v_3) \longmapsto v_1 - v_3,$$

which is a linear mapping from E^3 to R independent of whichever point p we choose. Hence this does give us a 1-form.

Now $\phi(U_1(p)) = \phi((1, 0, 0)_p) = 1$

$\phi(U_2(p)) = \phi((0, 1, 0)p) = 0$

$\phi(U_3(p)) = \phi((0, 0, 1)_p) = -1$

and hence, since $\phi = \sum_i \phi(U_i)dx_i$,

$$\phi = 1.dx_1 - 1.dx_3 = dx_1 - dx_3.$$

(b) This is not linear, since, for a fixed point p with $p_1 \neq p_3$ and non-zero vector v, the definition gives

$$\phi(2v_p) = \phi(v_p) = p_1 - p_3 \neq 0$$

while we would expect $\phi(2v_p) = 2\phi(v_p)$ if ϕ were linear.

(c) For fixed $p = (p_1, p_2, p_3)$ the mapping is

$$(v_1, v_2, v_3) \longmapsto p_3v_1 + p_1v_2,$$

which is linear and hence ϕ is a 1-form.

Now $\phi(U_1(p)) = \phi((1, 0, 0)_p) = p_3 = x_3(p)$ and hence $\phi(U_1) = x_3$;

$\phi(U_2(p)) = \phi((0, 1, 0)_p) = p_1 = x_1(p)$ and hence $\phi(U_2) = x_1$;

$\phi(U_3(p)) = \phi((0, 0, 1)_p) = 0$

and hence $\phi = \sum_i \phi(U_i)dx_i = x_3dx_1 + x_1dx_2$.

(d) For fixed p the mapping $v_p \longmapsto v_p[x^2+y^2]$ is linear, by Theorem I.3.3.(1), and since $v_p[x^2+y^2]$ is a real number ϕ_p is a linear functional and ϕ is a 1-form. Now by the definition of a differential

$$v_p[x^2+y^2] = d(x^2+y^2)(v_p).$$

Hence, since $\phi : v_p \longmapsto v_p[x^2+y^2] = d(x^2+y^2)(v_p)$, it follows that $\phi = d(x^2+y^2) = 2x\,dx + 2y\,dy$.

(e) This mapping is linear for each fixed p and so does define a 1-form. Obviously it is the zero 1-form, for which again we use the greatly overworked symbol 0.

(f) This fails to five us a 1-form for much the same reason as for part (b).

2. Page 25, Exercise 1(c).

If $\psi = \sum_i f_i dx_i$ then $\psi(v_p) = \sum f_i(p)v_i$.

Here $\psi = (z^2 - 1)dx - dy + x^2dz$ and hence

$$\psi(v_p) = (p_3^2 - 1)v_1 - v_2 + p_1^2v_3.$$

If $v_p = (1, 2, 3)\ (0, -2, 1)$ then $\psi(v_p) = -2$.

3. Page 25, Exercise 3.

$$\begin{aligned}\phi(W) &= (x^2dx - y^2dz)((xy + yz)U_1 - yzU_2 - xyU_3)\\ &= x^2(xy + yz)dx(U_1) - x^2yz\,dx(U_2) - x^3y\,dx\,(U_3)\\ &\quad -y^2(xy + yz)dz(U_1) + y^3z\,dz(U_2) + xy^3\,dz(U_3), \qquad \text{by linearity,}\\ &= x^2(xy + yz) + xy^3 = x^3y + x^2yz + xy^3\\ &= xy(x^2 + xz + y^2),\end{aligned}$$

since $dx(U_1) = dz(U_3) = 1$ and all the other similar terms are zero.

4. Page 26, Exercise 5(a).

By Corollary I.5.5, $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$

Here $f = (x^2 + y^2 + z^2)^{\frac{1}{2}}$ and hence $\frac{\partial f}{\partial x} = x(x^2 + y^2 + z^2)^{-\frac{1}{2}}.$

Using similar results for $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ we obtain

$$df = (x^2 + y^2 + z^2)^{-\frac{1}{2}}(x\,dx + y\,dy + z\,dz).$$

5. Page 26, Exercise 5(b).

When dealing with the function $\tan^{-1}(y/x)$ we must restrict ourselves to a region of E^3 on which the function x is never zero. The largest such domain is $\{p \in E^3 : p_1 \neq 0\}$.

Now $d(\tan^{-1}(y/x)) = (\tan^{-1})'(y/x)d(y/x)$, by Lemma I.5.7,

$$= \frac{1}{1+(y/x)^2}\,d(y/x),$$

$$= \frac{x^2}{x^2+y^2}\left(\frac{1}{x}\,dy + y\,d\left(\frac{1}{x}\right)\right),$$ by Lemma I.5.6,

$$= \frac{x^2}{x^2+y^2}\left(\frac{dy}{x} - \frac{y\,dx}{x^2}\right),$$ by Lemma I.5.7,

$$= \frac{x\,dy - y\,dx}{x^2+y^2}.$$

6. Page 26, Exercise 6(b).
By Corollary I.5.5

$$d(xe^{yz}) = \frac{\partial(xe^{yz})}{\partial x}dx + \frac{\partial(xe^{yz})}{\partial y}dy + \frac{\partial(xe^{yz})}{\partial z}dz$$

$$= e^{yz}(dx + xz\,dy + xy\,dz).$$

Hence $df[v_p] = e^{p_2p_3}(v_1 + p_1p_3v_2 + p_1p_2v_3)$ and if $v_p = (1, 2, 3)\ (0, -2, 1)$ it follows that $df[v_p] = e^{-2}$.

I.6 DIFFERENTIAL FORMS

Introduction

This section follows on from Section I.5, where we gave an abstract definition of a 1-form and then proved that any 1-form is a combination of the differentials of the natural coordinate functions. In this section we introduce 0-, 1-, 2- and 3-forms as formal expressions in terms of these differentials and define the wedge product and exterior derivative of such forms. In Chapter IV and Chapter VI we shall see that these higher dimensional forms can be defined in terms of multilinear functionals operating on tangent vectors.

READ: Section I.6 (pages 26-31).

Comments

(i) *Page 27: lines 15 to 19* Take these formal expressions as the definitions of 0-, 1-, 2-, 3-forms and treat the previous paragraphs as motivation only.

In certain cases it is possible to simplify the formal expression for a p-form without introducing any ambiguity.

(a) If some, but not all, of the coefficient functions are the zero functions we can omit the corresponding part of the linear combination. For example

$$e^x\,dx + xyz\,dy + 0dz \equiv e^x\,dx + xyz\,dy$$

and

$$z^2\,dx\,dy + 0\,dx\,dz + 0\,dy\,dz \equiv z^2\,dx\,dy.$$

(b) If all the coefficient functions are zero we have the zero p-form and when no confusion can arise we just write this as 0.

That is,

(i) the constant function with value 0 is the zero 0-form.

(ii) $0\,dx + 0\,dy + 0\,dz \equiv 0$ is the zero 1-form.

(iii) $0\,dx\,dy + 0\,dx\,dz + 0\,dy\,dz \equiv 0$ is the zero 2-form.

(iv) $0\,dx\,dy\,dz \equiv 0$ is the zero 3-form.

It is very unlikely that this abuse of the symbol 0 will ever lead to an ambiguity.

(c) If some of the coefficient functions are the constant function with value 1 we omit them. For example,

$$e^y\,dx + 1\,dy + 1\,dz = e^y\,dx + dy + dz$$

and $$e^y\,dx\,dy + 0\,dx\,dy + 1\,dy\,dz = e^y\,dx\,dy + dy\,dz.$$

(d) If some of the coefficient functions are the constant function with value −1 we omit the symbol "1". For example,

$$1\,dx\,dy + (-1)\,dx\,dz + 0dy\,dz \equiv dx\,dy - dx\,dz$$

and $$(-1)\,dx\,dy\,dz \equiv -dx\,dy\,dz.$$

More generally, if some coefficient function is $-f$, for some more commonly used function f, we write . . . $-f$ . . . instead of . . . + $(-f)$ . . . . For example,

$$x\,dx + (-y)\,dy + 0\,dz \equiv x\,dx - y\,dy.$$

(ii) *Page 27: line 23* Using the interpretation of 1-forms given in the last section we can prove that to add 1-forms or multiply them by some real-valued function all we have to do is to add the coefficient functions or multiply them by the given function. That is,

$$\Sigma f_i dx_i + \Sigma g_i dx_i = \Sigma(f_i + g_i)dx_i,$$
$$f(\Sigma f_i dx_i) = \Sigma(ff_i)dx_i.$$

Since we have no way of interpreting 2- and 3-forms yet the best we can do is to define addition and scalar multiplication in a similar way. That is

$$(f_1\,dx\,dy + g_1\,dx\,dz + h_1\,dy\,dz) + (f_2\,dx\,dy + g_2\,dx\,dz + h_2\,dy\,dz)$$
$$= (f_1 + f_2)dx\,dy + (g_1 + g_2)dx\,dz + (h_1 + h_2)dy\,dz;$$
$$k(f\,dx\,dy + g\,dx\,dy + h\,dy\,dz) = kf\,dx\,dy + kg\,dx\,dz + kh\,dy\,dz;$$
$$f_1\,dx\,dy\,dz + f_2\,dx\,dy\,dz = (f_1 + f_2)dx\,dy\,dz;$$
$$g(f\,dx\,dy\,dz) = gf\,dx\,dy\,dz.$$

(iii) *Page 27: Example 6.1* This example serves as a definition of the wedge product.

The product is given by the following procedure.

(a) Expand using the distributive law, bringing the coefficient functions to the front, dropping the symbol $\wedge$ but maintaining the order of the differentials.

(b) Drop terms with a repeated differential.

(c) Reorder the strings of differentials using the alternation rule.

(iv) *Page 28: Definition 6.3* The wedge products $df_i \wedge dx_i$ are expanded using Corollary I.5.5.

That is,

$$df_i \wedge dx_i = \left(\frac{\partial f_i}{\partial x_1} dx_1 + \frac{\partial f_i}{\partial x_2} dx_2 + \frac{\partial f_i}{\partial x_3} dx_3\right) \wedge dx_i$$
$$= \frac{\partial f_i}{\partial x_1} dx_1 dx_i + \frac{\partial f_i}{\partial x_2} dx_2 dx_i + \frac{\partial f_i}{\partial x_3} dx_3 dx_i.$$

The definition of the exterior derivative can be extended to cover 2-forms. If

$$\eta = f\,dx\,dy + g\,dx\,dz + h\,dy\,dz$$

then

$$d\eta = df \wedge dx\,dy + dg \wedge dx\,dz + dh \wedge dy\,dz$$

and this expression is again expanded using Corollary I.5.5.

(v) *Page 29: line 13* Though we can multiply 1-forms by any real-valued functions, in this formula the coefficients a and b are only real numbers or equivalently constant functions.

(vi) *Page 30: lines 11 and 12* Since dx dy dz is obtained from dz dx dy by two interchanges the expression is multiplied by $(-1)^2 = 1$.

Summary

Notation

f dx + g dy + h dz	Page 27, line 16
f dx dy + g dx dz + h dy dz	Page 27, line 18
f dx dy dz	Page 27, line 19
$\phi \wedge \psi$	Page 27, Example 6.1
$d\phi$	Page 28, Definition 6.3

Definitions

(i)	Differential form 0-form 1-form 2-form 3-form	Page 27, lines 15-19
(ii)	Addition and multiplication of forms	Page 27, lines 20-23
(iii)	Wedge product $\phi \wedge \psi$	Page 27, Example 6.1
(iv)	Exterior derivative $d\phi$	Page 28, Definition 6.3 and Text: page 38

Results

(i) The wedge product of 1-forms satisfies the alternation rule $\phi \wedge \psi = \psi \wedge \phi$ — Page 28, Lemma 6.2

(ii) The exterior derivative satisfies linear and Leibnizian properties

$$d(a\phi + b\psi) = a\,d\phi + b\,d\psi$$

$$d(fg) = df\,g + f\,dg$$

$$d(f\phi) = df \wedge \phi + f\,d\phi$$

$$d(\phi \wedge \psi) = d\phi \wedge \psi - \phi \wedge d\psi.$$

Page 29, line 13 and Theorem 6.4.

Techniques

(i)	Addition and scalar multiplication of differential forms.	Page 27, lines 20-23
(ii)	Expansion of the wedge product of differential forms.	Page 27, Example 6.1
(iii)	Expansion of the exterior derivative.	Page 28, Definition 6.3
(iv)	Expansion of the exterior derivatives using the linear and Leibnizian properties.	Page 29, Theorem 6.4

Exercises

Technique (i)

1. (a) Add the 2-forms

$$x\,dx\,dy + y\,dx\,dz \text{ and } -x\,dx\,dy + e^{xy}dx\,dz + 2dy\,dz.$$

(b) Multiply the second of the above 2-forms by the function e^{-xy}.

Technique (ii)

2. Form the wedge product of

(a) $yz\,dx + dz$ and $dy + z\,dz$,

(b) $yz\,dx + dz$ and $-x\,dx\,dy + e^{xy}dx\,dz + 2dy\,dz$.

Technique (iii)

3. Find the exterior derivative of

(a) $xyz\,dx + dz$,

(b) $xyz\,dx\,dy$.

Technique (iv)

4. Page 31, Exercise 3.

Theory Exercise

5. Page 31, Exercise 7.

Solutions

1. (a)

$$\begin{aligned}&(x\,dx\,dy + y\,dx\,dz) + (-x\,dx\,dy + e^{xy}dx\,dz + 2dy\,dz)\\ &= (x + (-x))dx\,dy + (y + e^{xy})dx\,dz + 2dy\,dz\\ &= 0\,dx\,dy + (y + e^{xy})dx\,dz + 2dy\,dz\\ &= (y + e^{xy})dx\,dz + 2dy\,dz.\end{aligned}$$

(b) $e^{-xy}(-x\,dx\,dy + e^{xy}dx\,dz + 2dy\,dz)$

$= -xe^{-xy}dx\,dy + e^{xy}e^{-xy}dx\,dz + 2e^{-xy}dy\,dz$

$= -xe^{-xy}dx\,dy + 1\,dx\,dz + 2e^{-xy}dy\,dz$

$= -xe^{-xy}dx\,dy + dx\,dz + 2e^{-xy}dy\,dz.$

2. (a) $(yz\,dx + dz) \wedge (dy + z\,dz)$

$= yz\,dx\,dy + yz^2\,dx\,dz + dz\,dy + z\,dz\,dz,$ by the distributive rule,

$= yz\,dx\,dy + yz^2\,dx\,dz + dz\,dy,$ since $dzdz = 0,$

$= yz\,dx\,dy + yz^2\,dx\,dz - dy\,dz,$ by the alternation rule.

(b) $(yz\,dx + dz) \wedge (-x\,dx\,dy + e^{xy}dx\,dz + 2dy\,dz)$

$= -xyz\,dx\,dx\,dy + yze^{xy}dx\,dx\,dz + 2yz\,dx\,dy\,dz$

$-x\,dz\,dx\,dy + e^{xy}dz\,dx\,dz + 2dz\,dy\,dz,$ by the distributive law,

$= 2yz\,dx\,dy\,dz + x\,dz\,dx\,dy,$ since $dx\,dx\,ay = dx\,dx\,dz$

$= dz\,dx\,dz = dz\,dy\,dz = 0,$

$= (2yz - x)dx\,dy\,dz,$ since $dz\,dx\,dy = dx\,dy\,dz.$

3. (a) $d(xyz\,dx + dz) = d(xyz\,dx + 1\,dz)$

$= d(xyz) \wedge dx + d(1) \wedge dz$

$= d(xyz) \wedge dx,$

since the differential $d(1)$, of the constant function with value 1, is the zero 1-form and since $0 \wedge dz$ is the zero 2-form,

$= (yz\,dx + xz\,dy + xy\,dz) \wedge dx,$

by Corollary I.5.5,

$= yz\,dx\,dx + xz\,dy\,dx + xy\,dz\,dx,$

by the distributive rule,

$= xz\,dy\,dx + xy\,dz\,dx,$

since $dx\,dx = 0,$

$= -xz\,dx\,dy - xy\,dx\,dz,$

by the alternation rule.

(b) $d(xyz\,dx\,dy) = d(xyz) \wedge dx\,dy$

$= (yz\,dx + xz\,dy + xy\,dz) \wedge dx\,dy$

$= yz\,dx\,dx\,dy + xz\,dy\,dx\,dy + xy\,dx\,dx\,dy$

$= xy\,dz\,dx\,dy$

$= xy\,dx\,dy\,dz.$

4. Page 31, Exercise 3.

By Corollary I.5.5, $df = \sum_i \dfrac{\partial f}{\partial x_i} dx_i$.

By Definition I.6.3, $d(df) = \sum_i d\left(\dfrac{\partial f}{\partial x_i}\right) \wedge dx_i$.

Again using Corollary I.5.5 we have

$$d(df) = \sum_i \left(\sum_j \frac{\partial^2 f}{\partial x_i \, \partial x_j} \; dx_j\right) \wedge dx_i = \sum_i \sum_j \frac{\partial^2 f}{\partial x_i \, \partial x_j} \; dx_j dx_i.$$

There are nine terms in this sum, which we must show is zero. We know that the terms involving $dx_1 dx_1$, $dx_2 dx_2$ and $dx_3 dx_3$ are already zero. The remaining six terms occur in pairs such as

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} \; dx_2 dx_1 \; + \; \frac{\partial^2 f}{\partial x_2 \partial x_1} \; dx_1 dx_2.$$

By the alternation rule this is

$$\left(- \frac{\partial^2 f}{\partial x_2 \partial x_1} + \frac{\partial^2 f}{\partial x_1 \partial x_2}\right) dx_1 dx_2.$$

But we know that $\partial^2 f/\partial x_1 \partial x_2 = \partial^2 f/\partial x_2 \partial x_1$ for suitably differentiable functions, and so the remaining terms in the sum for d(df) cancel in pairs.

Hence $d(df) = 0$.

By Theorem I.6.4,

$$\begin{aligned} d(f\,dg) &= df \wedge dg + f\,d(dg), \\ &= df \wedge dg, \qquad \text{by the above result.} \end{aligned}$$

5. Page 31, Exercise 7.

Any 1-form ϕ is of the form

$$\phi = \sum_i f_i dx_i,$$

where f_i and x_i are differentiable functions.

By Definition I.6.3,

$$d\varphi = \sum_i df_i \wedge dx_i$$

and by Theorem I.6.4 (3),

$$d(d\phi) = \sum_i (d(df_i) \wedge dx_i - df_i \wedge d(dx_i)).$$

By Exercise 3 we know that both $d(df_i)$ and $d(dx_i)$ are zero and so

$$d(d\phi) = 0.$$

I.7 MAPPINGS

Introduction

This section follows on from Sections I.3 and I.4.

In Section I.3 we introduced directional derivatives in order to differentiate functions from E^3 to R, and in Section I.4 we introduced the componentwise differentiation of functions from R to E^3. In this section we combine these ideas in order to differentiate functions from E^n to E^m, for any positive integers n and m. The derivative mappings of these functions turn out to be linear transformations between corresponding tangent spaces and we shall describe the matrices representing them.

READ: Section I.7 (pages 32-39).

Comments

(i) *Section I.7* In this section there are several straightforward generalizations of definitions that have previously been applied to E^3 only. Those that are not specifically mentioned are:

(a) Natural coordinate functions $x_1, \ldots, x_n$, which generalize those of Definition I.1.2. If p is a point in E^n then $x_i(p) = p_i$ $(i = 1, 2, \ldots, n)$.

(b) Tangent vectors v_p in E^n, which generalize those of Definition I.2.1. The tangent vector v_p represents the "arrow" from p to p + v in E^n.

(c) Curves $\alpha(t) = (\alpha_1(t), \ldots, \alpha_n(t))$ in E^n, which generalize those of Definition I.4.1.

(d) Straight lines $t \longmapsto p + tv$ in E^n.

(e) The velocity vector $\alpha'(t) = ((d\alpha_1/dt)(t), \ldots, (d\alpha_n/dt)(t))_{\alpha(t)}$ for a curve α in E^n, which generalizes Definition I.4.3.

(f) The directional derivative $v_p[f]$ of a function f on E^n with respect to a tangent vector v_p, which generalizes that of Definition I.3.1. That is, $v_p[f] = (d/dt)\,(f(p + tv))\big|_{t=0}$.

(ii) *Page 36: line 3*

$$\begin{aligned}
F_*(v) &= F(p + tv)'(0) \\
&= ((p_1 + tv_1)^2 - (p_2 + tv_2)^2, 2(p_1 + tv_1)(p_2 + tv_2))'(0) \\
&= (2(p_1 + tv_1)v_1 - 2(p_1 + tv_2)v_2, 2(p_1 + tv_1)v_2 + 2v_1(p_2 + tv_2))(0) \\
&= 2(p_1v_1 - p_2v_2, p_1v_2 + p_2v_1).
\end{aligned}$$

(iii) *Page 37: following Corollary 7.7* To obtain the j-th column of a matrix representing a linear transformation with respect to given bases, we find the coordinates for the image of the j-th basis vector in the domain, in terms of the basis for the codomain.

In Example 7.3(2) we take as basis vectors for the domain $U_1(p)$ and $U_2(p)$ and for the codomain $U_1(F(p))$ and $U_2(F(p))$. Hence, by Corollary 7.7, the linear transformation is represented by

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial u}(p) & \dfrac{\partial f_1}{\partial v}(p) \\[2ex] \dfrac{\partial f_2}{\partial u}(p) & \dfrac{\partial f_2}{\partial v}(p) \end{pmatrix}$$

where $f_1 = u^2 - v^2$ and $f_2 = 2uv$.

All such matrices are called *Jacobian matrices.*

Hence at $p = (p_1, p_2)$ the Jacobian matrix is

$$\begin{pmatrix} 2u(p) & -2v(p) \\ 2v(p) & 2u(p) \end{pmatrix} = \begin{pmatrix} 2p_1 & -2p_2 \\ 2p_2 & 2p_1 \end{pmatrix}$$

The image of a typical vector $v = (v_1, v_2)$ under this transformation is

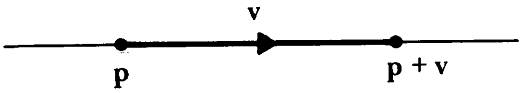
$$\begin{pmatrix} 2p_1 & -2p_2 \\ 2p_2 & 2p_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 2 \begin{pmatrix} p_1v_1 - p_2v_2 \\ p_2v_1 - p_1v_2 \end{pmatrix}$$

which is the answer obtained on page 36, line 3. Note that while dealing with matrices we represent tangent vectors by column vectors.

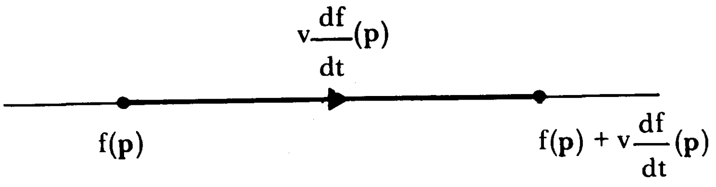
Additional Text

(a) Let f be a mapping from R to R. Then the Jacobian matrix is the 1×1 matrix (df/dt), where here we do not need partial derivatives since f is a function of t only.

Now a tangent vector in R, v_p is an arrow from p to p + v,



and this is mapped by f_{*p} to the tangent vector $v(df/dt)(p)$ at $f(p)$. That is it is multiplied in length by $(df/dt)(p)$.



(b) Suppose we have a curve $\alpha : R \longrightarrow R^3$, where α has coordinate functions $(\alpha_1, \alpha_2, \alpha_3)$, i.e.

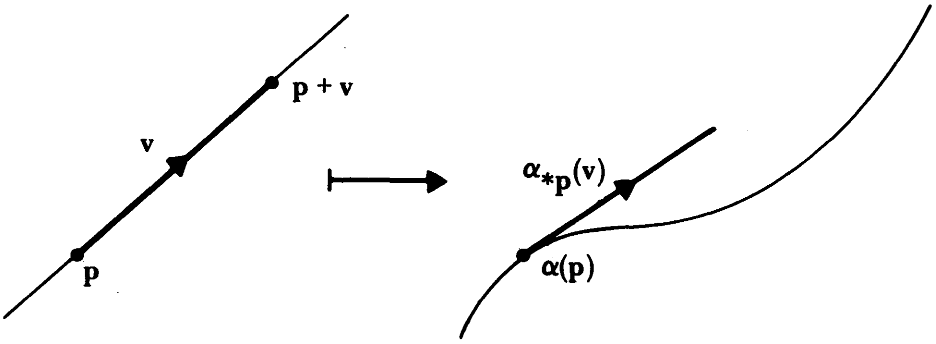
$$\alpha : t \longmapsto \alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t)).$$

Then we obtain the 3 X 1 Jacobian matrix

$$\begin{pmatrix} \dfrac{d\alpha_1}{dt} \\ \dfrac{d\alpha_2}{dt} \\ \dfrac{d\alpha_3}{dt} \end{pmatrix}$$

A tangent vector v, at p, is mapped to the tangent vector $\alpha_{*p}(v)$, at $\alpha(p)$. Reverting to row vectors, this is the tangent vector

$$\left(v\frac{d\alpha_1}{dt}(p),\quad v\frac{d\alpha_2}{dt}(p),\quad v\frac{d\alpha_3}{dt}(p)\right)_{\alpha(p)}$$



(c) Suppose we have a real-valued function $F : E^3 \longrightarrow R$, then the Jacobian matrix is the 1 X 3 matrix

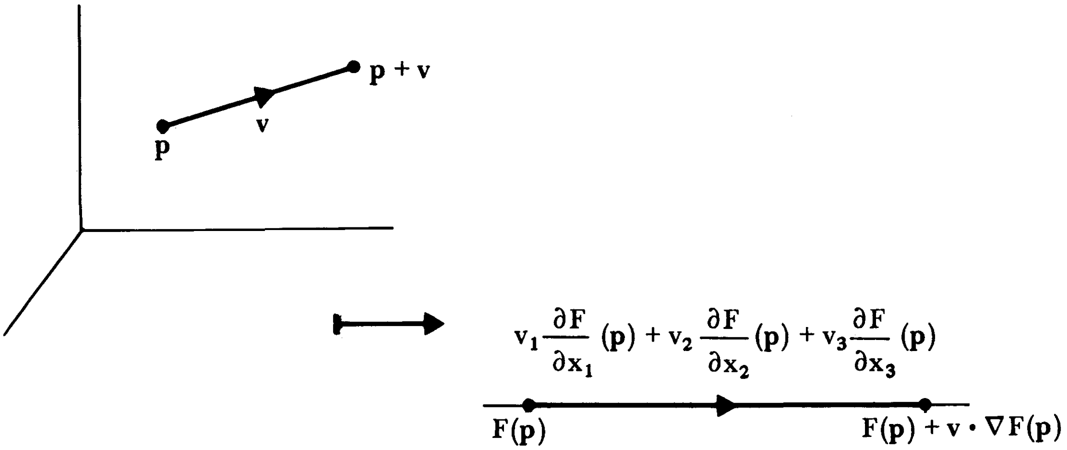
$$\left(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \frac{\partial F}{\partial x_3}\right).$$

If $v_p = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_p$ is a typical tangent vector, it is mapped by F_{*p}

to the tangent vector

$$\left(\frac{\partial F}{\partial x_1}(p), \frac{\partial F}{\partial x_2}(p), \frac{\partial F}{\partial x_3}(p)\right)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \text{ at } F(p) \text{ in } R.$$

This is $v_1\ (\partial F/\partial x_1)(p) + v_2\ (\partial F/\partial x_2)(p) + v_3\ (\partial F/\partial x_3)(p)$, which we have already met as the directional derivative $v_p[F]$.


p + v
v
p
$v_1 \frac{\partial F}{\partial x_1}(p) + v_2 \frac{\partial F}{\partial x_2}(p) + v_3 \frac{\partial F}{\partial x_3}(p)$
F(p)
F(p) + v • ∇F(p)

(d) *The Composite Rule* Suppose we have two mappings $F : E^n \longrightarrow E^m$ and $G : E^m \longrightarrow E^r$, then we can form the composite mapping $G \circ F : E^n \longrightarrow E^r$. The obvious question is whether we can calculate the derivative map $(G \circ F)_*$ in terms of the derivatives G_* and F_*. The answer is yes. We state the result, which should be intuitively obvious, and investigate some of its consequences.

Theorem

For mappings $F : E^n \longrightarrow E^m$ and $G : E^m \longrightarrow E^r$

$$(G \circ F)_* = G_* \circ F_*.$$

To be more precise we need to take into consideration the point at which each of the derivatives is evaluated.

Now

$$F_{*p} : T_p(E^n) \longrightarrow T_{F(p)}(E^m)$$

$$G_{*F(p)} : T_{F(p)}(E^m) \longrightarrow T_{G \circ F(p)}(E^r)$$

$$(G \circ F)_{*p} : T_p(E^n) \longrightarrow T_{G \circ F(p)}(E^r)$$

and hence $(G \circ F)_{*p} = G_{*F(p)} \circ F_{*p}$. (This may remind you of the composite rule for ordinary differentiation – $D(g \circ f) = (Dg \circ f).Df$.)

Considering all points p of the domain at the same time we can write the composite rule as

$$(G \circ F)_* = G_{*F} \circ F_*.$$

This form of the composite rule accounts for all of the composite rules that we have encountered. When we expand it in terms of Jacobian matrices the composition on the right hand side then stands for the usual matrix multiplication.

(i) Given maps $f : R \longrightarrow R$ and $g : R \longrightarrow R$, the composite rule gives the form of the chain rule encountered in M100.

(ii) Given a map $h : R \longrightarrow R$ followed by $\alpha : R \longrightarrow E^3$, as in Lemma I.4.5, the composite rule gives

$$(\alpha \circ h)_{*s} = \alpha_{*h(s)} \circ h_{*s}.$$

Now $h_* = \dfrac{dh}{ds}$ and α_* is represented by $\begin{pmatrix} \dfrac{d\alpha_1}{dt} \\ \dfrac{d\alpha_2}{dt} \\ \dfrac{d\alpha_3}{dt} \end{pmatrix}$

Hence if we write $\beta = \alpha \circ h$ the composite rule gives

$$\begin{pmatrix} \dfrac{d\beta_1}{dt}(s) \\ \dfrac{d\beta_2}{dt}(s) \\ \dfrac{d\beta_3}{dt}(s) \end{pmatrix} = \begin{pmatrix} \dfrac{d\alpha_1}{dt}(h(s)) \\ \dfrac{d\alpha_2}{dt}(h(s)) \\ \dfrac{d\alpha_3}{dt}(h(s)) \end{pmatrix} \cdot \frac{dh}{ds}(s)$$

i.e. $\beta'(s) = \dfrac{dh}{ds}(s)\alpha'(s)$.

(iii) Given a map $f : E^3 \longrightarrow R$ followed by $h : R \longrightarrow R$, as in Lemma I.5.7, we can write the composite rule as

$$(h \circ f)_* = h_{*f} \circ f_*.$$

Now h_* is just the derivative which can be written h', and so $h_{*f} = h'(f)$. Hence in terms of Jacobian matrices

$$\begin{pmatrix} \dfrac{\partial(h(f))}{\partial x_1}, & \dfrac{\partial(h(f))}{\partial x_2}, & \dfrac{\partial(h(f))}{\partial x_3} \end{pmatrix} = h'(f)\begin{pmatrix} \dfrac{\partial f}{\partial x_1}, & \dfrac{\partial f}{\partial x_2}, & \dfrac{\partial f}{\partial x_3} \end{pmatrix},$$

i.e. $\dfrac{\partial(h(f))}{\partial x_i} = h'(f)\dfrac{\partial f}{\partial x_i}$ $(i = 1,2,3)$.

(iv) Given a map $\alpha : R \longrightarrow E^3$ followed by a map $f : E^3 \longrightarrow R$, as in Lemma I.4.6, then the composite rule states

$$(f \circ \alpha)_{*t} = f_{*\alpha(t)} \circ \alpha_{*t}.$$

In terms of Jacobian matrices, this becomes

$$\frac{d(f(\alpha))}{dt}(t) = \left(\frac{\partial f}{\partial x_1}(\alpha(t)),\ \frac{\partial f}{\partial x_2}(\alpha(t)),\ \frac{\partial f}{\partial x_3}(\alpha(t))\right)\begin{pmatrix}\dfrac{d\alpha_1}{dt}(t)\\[2ex] \dfrac{d\alpha_2}{dt}(t)\\[2ex] \dfrac{d\alpha_3}{dt}(t)\end{pmatrix}$$

so

$$\frac{d(f(\alpha))}{dt}(t) = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i}(\alpha(t))\frac{d\alpha_i}{dt}(t).$$

Now read the proof of Lemma I.4.6 and Lemma I.3.2, where $\alpha : t \longmapsto p + tv$, so that $(d\alpha_i/dt)(0) = (d/dt)(p_i + tv_i)\big|_{t=0}\, v_i$.

Summary

Notation

$F = (f_1, \ldots, f_m)$	Page 33, Definition 7.1
F_*	Page 35, Definition 7.4
$\begin{pmatrix}\dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial v}\\[2ex] \dfrac{\partial g}{\partial u} & \dfrac{\partial g}{\partial v}\end{pmatrix}$	Page 37 and Text, page 44

Definitions

(i)	Euclidean coordinate functions $(f_1, \ldots, f_m)$	Page 33, Definition 7.1
(ii)	Image of curve under a mapping $F(\alpha)$	Page 33, Definition 7.2
(iii)	The derivative map F_*	Page 35, Definition 7.4
(iv)	Jacobian matrix	Page 37 and Text, page 44

Results

(i)	The derivative map can be described in terms of the coordinate functions and directional derivatives $F_*(v) = (v[f_1], \ldots, v[f_m])$.	Page 36, Theorem 7.5
(ii)	The derivative map $F_{*p} : T_p(E^n) \longrightarrow T_{F(p)}(E^m)$ is a linear transformation.	Page 36, Corollary 7.6
(iii)	The effect of F_* on the standard basis is $F_*(U_j) = \sum_i \frac{\partial f_i}{\partial x_j} \overline{U}_i$.	Page 37, Corollary 7.7

(iv) The derivative map preserves velocities.

If $\beta = F(\alpha)$ then $\beta' = F_*(\alpha')$.
Page 38, Theorem 7.8

(v) The composite rule:

If $F : E^n \longrightarrow E^m$ and $G : E^m \longrightarrow E^r$ then

$(G \circ F)_{*p} = G_{*F(p)} \circ F_{*p}$.

Techniques

(i) Description of mappings in terms of Euclidean coordinate functions.
Page 33, Definition 7.1

(ii) Pictorial description of simple mappings.

(iii) Determination of derivative maps,

- (a) from first principles
  Page 35, Definition 7.4
- (b) from directional derivatives of coordinate functions
  Page 36, Theorem 7.5
- (c) from Jacobian matrix.
  Text, page 44

(iv) Calculation of the derivative mapping of a composite using the composite rule:

$$(G \circ F)_* = G_{*F} \circ F_*$$

where if the derivatives are represented by Jacobian matrices the composition on the right is matrix multiplication.

Exercises

Technique (i)

1. Express the following mappings from E^3 to E^3 in terms of Euclidean coordinate functions

   (a) $F : p \longmapsto -3p$,

   (b) $F : p \longmapsto (e^{p_1 p_2}, p_3 + 2p_1, p_1^2)$.

Technique (ii)

2. Page 39, Exercise 2.

Technique (iii)

3. (a) Page 39, Exercise 4.

   (b) Find $F_*(v_p)$, using directional derivatives, for $F = (x \cos y, x \sin y, z)$, $v = (2, -1, 3)$ and $p = (0, 0, 0)$.

   (c) Describe the derivative mappings for F given in Example (1), page 34.

Technique (iv)

4. Use the composite rule for Jacobian matrices to find $(\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$ for the composite mapping in Exercise 4(a), page 6.

Solutions

1. (a) Since $p \longmapsto -3p$,

$$F(p) = (-3p_1, -3p_2, -3p_3)$$
$$= (-3x(p), -3y(p), -3z(p))$$

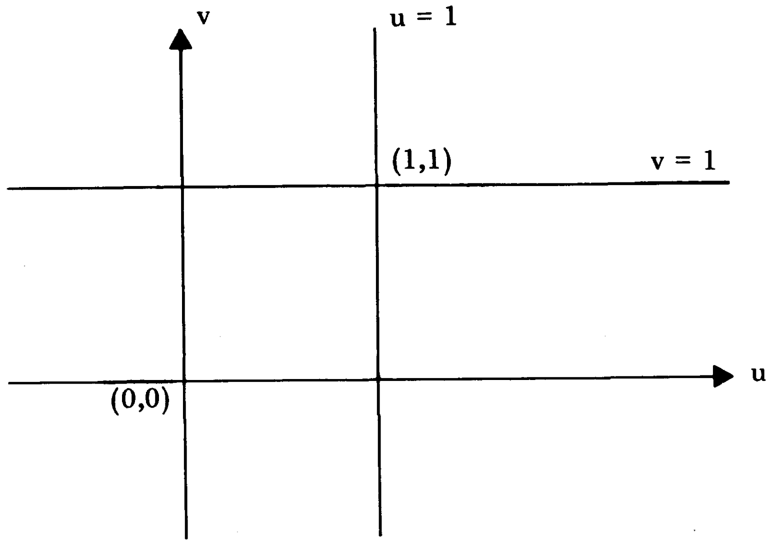
and so, in terms of Euclidean coordinate functions,

$$F = (-3x, -3y, -3z).$$

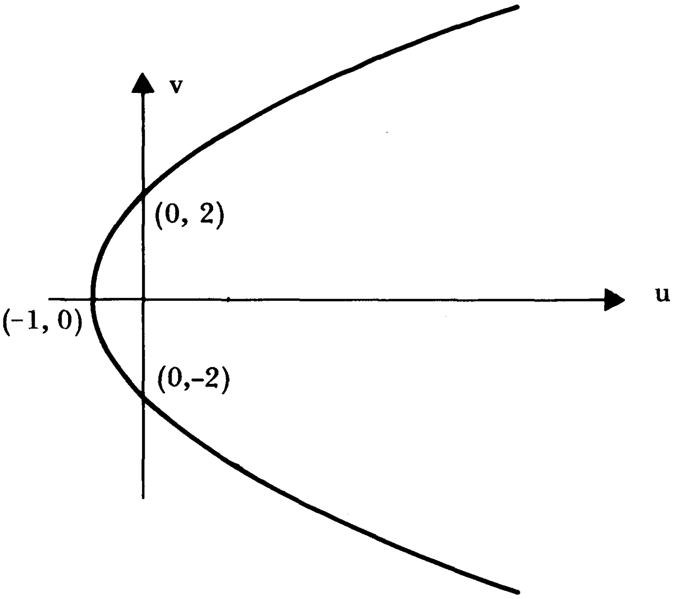
(b) $$F(p) = (e^{p_1 p_2}, p_3 + 2p_1, p_1^2)$$
$$= (e^{x(p)y(p)}, z(p) + 2x(p), (x(p))^2)$$
$$= (e^{xy}(p), (z + 2x)(p), x^2(p))$$

and so $F = (e^{xy}, z + 2x, x^2)$.

2. Page 39, Exercise 2.
The lines $u = 1$, and $v = 1$ are the sets of points p such that $u(p) = p_1 = 1$ and $v(p) = p_2 = 1$. These are shown on the following diagram.



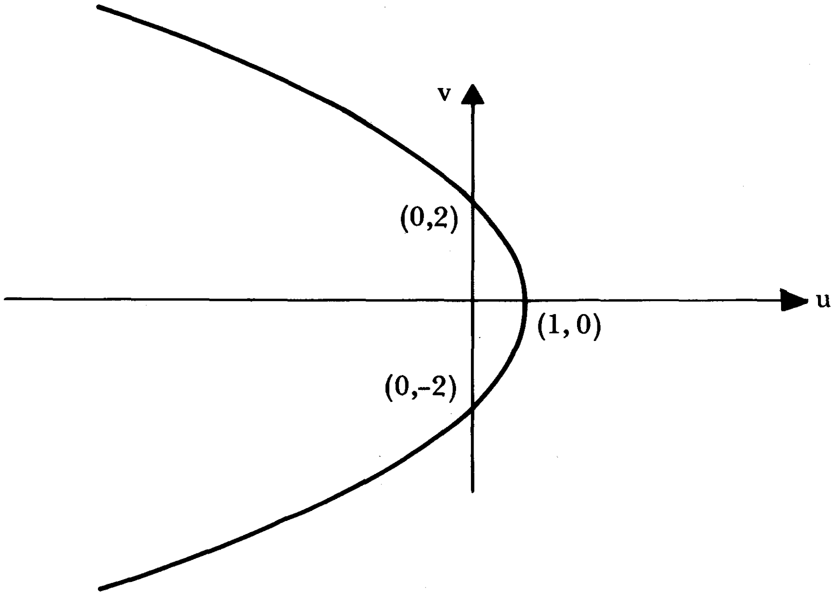
A point on the horizontal line is of the form $(t, 1)$ and is mapped by F to $(t^2 - 1, 2t)$. So, for instance, $(0, 1) \longmapsto (-1, 0)$, $(1, 1) \longmapsto (0, 2)$ and $(-1, 1) \longmapsto (0, -2)$.



How do we describe such a curve? A point $p = (p_1, p_2)$ is on the curve if $p_1 = t^2 - 1, p_2 = 2t$ for some t. From these identities we deduce that $4(p_1 + 1) = p_2^2$. Using the notation of Section I.4 we write

$$p \in M : 4(u + 1) - v^2 = 0.$$

The line $u = 1$ consists of points of the form $(1, t)$. These are mapped by F to $(1 - t^2, 2t)$. This is a typical point on the parabola $v^2 = -4(u - 1)$, which is shown on the following diagram.



3. (a) Page 39, Exercise 4(a).

$$F_*(v_p) = \frac{d}{dt}(F(p + tv))\Big|_{t=0} \text{ at } F(p)$$

$$= \frac{d}{dt}(F(p_1 + tv_1, p_2 + tv_2, p_3 + tv_3))\Big|_{t=0}$$

$$= \frac{d}{dt}(p_1 + tv_1 - p_2 - tv_2, p_1 + tv_1 + p_2 + tv_2, 2p_3 + 2tv_3)\Big|_{t=0}$$

$$= (v_1 - v_2, v_1 + v_2, 2v_3) \text{ at } F(p).$$

Page 39, Exercise 4(b).

If F is linear then $F(p + tv) = F(p) + tF(v)$.

$$\text{Hence } F_*(v_p) = \frac{d}{dt}(F(p + tv))|_{t=0}, \text{ at } F(p),$$

$$= \frac{d}{dt}(F(p) + tF(v))|_{t=0}, \text{ at } F(p),$$

$$= F(v) \text{ at } F(p),$$

so $F_*(v_p) = F(v)_{F(p)}$.

(b) If $F = (x\cos y,\ x\sin y,\ z)$, then by Theorem I.7.5

$F_*(v_p) = (v_p[x\cos y],\ v_p[x\sin y],\ v_p[z])_{F(p)}$.

Now if $v = (2,-1,3)$ and $p = (0,0,0)$, then by Lemma I.3.2

$$\begin{aligned} v_p[x\cos y] &= 2\frac{\partial}{\partial x}(x\cos y)(p) - \frac{\partial}{\partial y}(x\cos y)(p) + 3\frac{\partial}{\partial z}(x\cos y)(p) \\ &= 2\cos y(0,0,0) + x\sin y(0,0,0) + 3.0(0,0,0) \\ &= 2\cos 0 + 0\sin 0 + 3\cdot 0 \\ &= 2. \end{aligned}$$

Similarly,

$$\begin{aligned} v_p[x\sin y] &= 2\sin y(0,0,0) - x\cos y(0,0,0) + 0 \\ &= 2\sin 0 - 0\cos 0 = 0. \end{aligned}$$

$$\begin{aligned} v_p[z] &= 2.0 - 1.0 + 3.1 \\ &= 3. \end{aligned}$$

So

$$F_*(v_p) = (2,0,0)_{(0\cos 0,\ 0\sin 0,\ 0)} = (2,0,3)_{(0,0,0)}.$$

(c) If $F = (x - y, x + y, z)$ the Jacobian matrix for F_* is

$$\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This maps the vector v to

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 - v_2 \\ v_1 + v_2 \\ v_3 \end{pmatrix}$$

and so $F_*(v_p) = (v_1 - v_2, v_1 + v_2, v_3)_{F(p)}$, as expected.

4. We have f described as a composite of functions, $f = h \circ g$, where $g : p \longmapsto (g_1(p), g_2(p), g_3(p))$. That is, g_1, g_2, g_3 are the coordinate functions for g and so in this example

$$g = (x + y, y^2, x + z).$$

Since $f : E^3 \longrightarrow R$, $g : E^3 \longrightarrow E^3$ and $h : E^3 \longrightarrow R$, the corresponding Jacobian matrices are

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right),$$

$$\begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} & \frac{\partial g_1}{\partial z} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} & \frac{\partial g_2}{\partial z} \\ \frac{\partial g_3}{\partial x} & \frac{\partial g_3}{\partial y} & \frac{\partial g_3}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2y & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

and $\left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z}\right) = (2x, -z, -y)$.

Now by the composite rule $f_{*p} = h_{*g(p)} \circ g_{*p}$

so

$$\left(\frac{\partial f}{\partial x}(p), \frac{\partial f}{\partial y}(p), \frac{\partial f}{\partial z}(p)\right) = \Big(2x(g(p)), -z(g(p)), -y(g(p))\Big) \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2y(p) & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \Big(2g_1(p), -g_3(p), -g_2(p)\Big) \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2p_2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \left(2p_1 + 2p_2, -p_1 - p_3, -p_2^2\right) \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2p_2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= (2p_1 + 2p_2 - p_2^2, 2p_1 + 2p_2 - 2p_1p_2 - 2p_2p_3, -p_2^2),$$

and hence

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2x + 2y - y^2, 2x + 2y - 2xy - 2yz, -y^2).$$

FURTHER EXERCISES AND SOLUTIONS

Section I.1

Technique Exercises

Pages 5-6, Exercises 1(b), 2(a), 2(d), 4(b), 4(c).

Solutions

1(b). $xy(2y + x)\sin z$

2(a). 0

2(d). $t^4(1 - t^3)$

4(b). $\dfrac{\partial f}{\partial x} = 2e^{2x}(1 - e^y)$

4(c). $\dfrac{\partial f}{\partial x} = 4x.$

Section I.2

Technique Exercise

Page 11, Exercise 4.

Other Recommended Exercise

Page 11, Exercise 5. This extends the concepts of linear independence and linear combinations to cover vector fields.

Solutions

4. $f = kx^2$, $g = -ky^2$ for any real function k on R^3,

e.g. $f = x^2$, $g = -y^2$.

5(a). $V_1(p) = (1,0,-p_1)_p$

$V_2(p) = (0,1,0)_p$

$V_3(p) = (p_1,0,1)_p$

and since $\begin{vmatrix} 1 & 0 & p_1 \\ 0 & 1 & 0 \\ p_1 & 0 & 1 \end{vmatrix} = 1 + p_1^2 \neq 0$ these vector fields are linearly independent.

5(b).
$$\begin{aligned}(xU_1 + yU_2 + zU_3)(p) &= (p_1,p_2,p_3)_p \\ &= \frac{p_1(1 - p_3)}{(1 + p_1^2)} V_1(p) + p_2 V_2(p) + \frac{(p_1^2 + p_3)}{(1 + p_1^2)} V_3(p)\end{aligned}$$

and hence

$$xU_1 + yU_2 + zU_3 = \frac{x(1 - z)}{(1 + x^2)} V_1 + yV_2 + \frac{(x^2 + z)}{(1 + x^2)} V_3.$$

Section I.3

Technique Exercises

Pages 14-15, Exercises 1(a), (b), 2(a), (b), 3(a)-(d), (f).

Solutions

1(a) and 2(a). 0

1(b) and 2(b). 896

3(a). y^3

3(b). $-3xz^2$

3(c). $yz^2(y^2z - 3x^2)$

3(d). $-yz^2(y^2z - 3x^2)$

3(f). 0

Section I.4

Technique Exercise

Page 21, Exercise 8.

Other Recommended Exercises

Page 21, Exercise 4. This exercise shows that a curve is determined uniquely by its initial point and velocity vectors.

Page 21, Exercise 6. This theory exercise deals with the definition of the directional derivative.

Page 21, Exercise 9. This deals with the geometrical interpretation of tangent lines.

Solutions

8. $$\beta(s) = \left(s, \frac{1}{s}, \sqrt{2}\log s\right),$$

$$\beta'(s) = \left(1, -\frac{1}{s^2}, \frac{\sqrt{2}}{s}\right),$$

$$\alpha'(h(s)) = \left(s, -\frac{1}{s}, \sqrt{2}\right).$$

$\frac{dh}{ds}(s) = \frac{1}{s}$ and hence $\beta'(s) = \frac{dh}{ds}(s).\alpha'(h(s))$.

4. $$\alpha(t) = \left(1 + \frac{t^3}{3}, \frac{t^2}{2}, e^t - 6\right).$$

6. Let α be any curve with $\alpha'(0) = v_p$.

Then $\dfrac{d(f(\alpha))}{dt}(0) = \alpha'(0)[f] = v_p[f]$.

9. (a) $u \longmapsto (2, 2u, u)$.

(b) $u \longmapsto (\sqrt{2}(1-u), \sqrt{2}(1+u), \frac{\pi}{4} + u)$.

Section I.5

Technique Exercises

Pages 25-26, Exercises 1(a), (b), 3(a), (c), 4, 6(a), (c).

Other Recommended Exercises

Page 25, Exercise 2. This expresses the operation of a 1-form on a vector field in terms of Euclidean coordinate functions.

Page 26, Exercise 8. This deals with the Leibnizian property of d.

Page 26, Exercises 9, 10. These explore the relationship between the differential of a function and its maxima and minima.

Page 26, Exercise 11. This deals with the relationship between the differential and the difference operator.

Solutions

1(a). 4

1(b). -4

3(a). $x^3 + x^2(1 + z) - xy^2 - \dfrac{y^2 z}{x}$

4(a). $5f^4 df$

4(b). $\dfrac{df}{2\sqrt{f}}$

4(c). $\dfrac{2f\,df}{1 + f^2}$

6(a). $df = y^2 dx + (2xy - z^2)dy - 2yz\,dz$, -10.

6(c). $df = \cos(xy)\cos(xz)(y\,dx + x\,dy) - \sin(xy)\sin(xz)(z\,dx + x\,dz)$, -2.

2. $\phi(V) = \sum_i \sum_j f_i v_j\, dx_i(U_j) = \sum_i \sum_j f_i v_j \delta_{ij} = \sum_i f_i v_i$.

8. $$\begin{aligned} d(fg)(v_p) &= v_p[fg] = v_p[f]g + fv_p[g] \\ &= df(v_p)\, g + f\, dg(v_p) \\ &= (g\,df + f\,dg)(v_p) \end{aligned}$$

and hence $d(fg) = g\,df + f\,dg$.

9. $$df(v_p) = \frac{\partial f}{\partial x}(p)v_1 + \frac{\partial f}{\partial y}(p)v_2 + \frac{\partial f}{\partial z}(p)v_3$$

and hence $df(v_p) = 0$ for all **v** if and only if

$$\frac{\partial f}{\partial x}(p) = \frac{\partial f}{\partial y}(p) = \frac{\partial f}{\partial z}(p) = 0.$$

Here $\frac{\partial f}{\partial x} = -2xy, \frac{\partial f}{\partial y} = 1 - x^2 - 2yz, \frac{\partial f}{\partial z} = 1 - y^2$

and the critical points are $\pm\,(0, 1, \frac{1}{2})$.

10. If p is a local maximum or minimum then $t = 0$ is a local maximum or minimum of the function $t \longmapsto f(p + tv)$.

Hence $df[v_p] = v_p[f] = \frac{d}{dt}(f(p + tv))\,|_{t=0} = 0$.

11(a). By Taylor's Theorem

$$f(p + tv) = f(p) + t\frac{d}{dt}(f(p + tv))\,|_{t=0} + \text{Remainder}$$

$$= f(p) + t\,df(v_p) + R.$$

Hence $f(p + v) - f(p) \sim df(v_p)$.

11(b). Exact: $-0{\cdot}420$

Approximate: $df(v_p) = \left(\frac{2xy}{z}\,dx + \frac{x^2}{z}\,dy - \frac{x^2y}{z^2}\,dz\right)(-0{\cdot}1, 0{\cdot}1, 0{\cdot}2)_{(1,\ 1{\cdot}5,\ 1)}$

$$= -\tfrac{1}{2}.$$

Section I.6

Technique Exercises

Page 31, Exercises 1, 2, 4(a), (b), (d), 6.

Other Recommended Exercises

Page 31, Exercise 5. This expresses the wedge product of three 1-forms in terms of the determinant of their Euclidean coordinates.

Page 31, Exercise 8. This exercise deals with the relationship between the exterior derivative and the vector operations div, grad and curl.

Solutions

1(a). $yz \cos z\, dx\, dy - \sin z\, dx\, dz - \cos z\, dy\, dz$,

$\sin z\, dx\, dy + z \sin z\, dx\, dz + z \cos z\, dy\, dz$,

$-yz\, dx\, dy - yz^2\, dx\, dz + dy\, dz$.

1(b). $-z\,dx\,dy - y\,dx\,dz$,

$-\cos z\,dx\,dz + \sin z\,dy\,dz$,

0.

2. $d\phi = \frac{1}{y^2}\,dx\,dy,\ d\psi = -\,dy\,dz$,

$\phi\wedge\psi = \frac{z}{y}\,dx\,dy$ and

$d(\phi\wedge\psi) = d\phi\wedge\psi - \phi\wedge d\psi = \frac{1}{y}\,dx\,dy\,dz$.

4(a). $d(f\,dg + g\,df) = d(d(fg)) = 0$.

4(b). $(df - dg)\wedge(df + dg) = 2\,df\wedge dg$.

4(d). $(1 - f)df\wedge dg$

6. $dx\,dy\,dz = r\,dr\,d\vartheta\,dz$

5. Expand both the determinant and the triple wedge product in terms of the Euclidean coordinate functions. Both sides give

$$(f_{11}f_{22}f_{33} - f_{11}f_{23}f_{32} + f_{12}f_{23}f_{31} - f_{12}f_{21}f_{33} + f_{13}f_{21}f_{32} - f_{13}f_{22}f_{31})dx\,dy\,dz.$$

8(a). $df = \Sigma\frac{\partial f}{\partial x_i}\,dx_i \xleftrightarrow{(1)} \Sigma\frac{\partial f}{\partial x_i}\,U_i = \text{grad } f.$

8(b). If $V = \Sigma f_i\,U_i$ then $\phi = \Sigma f_i\,dx_i$,

$$d\phi = \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)dx_1dx_2 + \left(\frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3}\right)dx_1dx_3 + \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right)dx_2dx_3$$

and $d\phi \xleftrightarrow{(2)}$ curl V.

8(c). If $V = \Sigma f_iU_i$ then $\eta = f_3dx_1dx_2 - f_2dx_1dx_3 + f_1dx_2dx_3$.

$d\eta = \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\right)dx_1\,dx_2\,dx_3$ and hence $d\eta \xleftrightarrow{(2)}$ (div V)$dx\,dy\,dz$.

Section I.7

Technique Exercise

Page 39, Exercise 1.

Other Recommended Exercises

Page 40, Exercises 5 and 6. These describe the derivative of a mapping in terms of the differentials of its Euclidean coordinate functions.

Page 40, Exercise 8. This deals with the definition of the derivative.

Page 40, Exercise 9. The derivative mapping preserves directional derivatives.

Page 40, Exercise 11. This deals with inverse mappings, which you can read about in Exercise 10.

Page 40, Exercises 7 and 12. These deal with the derivative of a composite mapping.

Solutions

1(a). $(0, 0)$.

1(b). $(-3,1), (3,-1)$.

1(c). $(0, 0), (1, 0)$.

5. $F_* = (\cos y\, dx - x \sin y\, dy, \sin y\, dx + x \cos y\, dy, dz)$

(a) $(2,0,3)_{(0,0,0)}$

(b) $(2,2,3)_{(0,2,\pi)}$

6. The Jacobian matrix at p is

$$\begin{pmatrix} \cos p_2 & \sin p_2 & 0 \\ -p_1 \sin p_2 & p_1 \cos p_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This has determinant p_1 and hence F is not regular at points for which $p_1 = 0$.

8. Let α be any curve such that $\alpha'(0) = v_p$, then if $\beta = F(\alpha)$ it follows that

$$\beta'(0) = F_*(\alpha'(0)) = F_*(v_p).$$

9. Let $\alpha : t \longmapsto p + tq$.

$$\begin{tikzcd} R \arrow[r, "\alpha"] & E^n \arrow[r, "F"] & E^m \arrow[r, "g"] & R \end{tikzcd}$$

$$v_p[g(F)] = \alpha'(0)[g(F)] = \frac{d}{dt}(g(F(\alpha)))\big|_{t=0}.$$

$$F_*(v_p) = (F(\alpha))'(0).$$

$$F_*(v_p)[g] = (F(\alpha))'(0)[g] = \frac{d}{dt}(g(F(\alpha)))\big|_{t=0}.$$

Hence $v_p[g(F)] = F_*(v_p)[g]$.

11(a). $F^{-1} = (v, ue^{-v})$. This is a diffeomorphism.

11(b). $F^{-1} = (u^{\frac{1}{3}}, v + u^{\frac{1}{3}})$. This is not a diffeomorphism since it is not differentiable when $u = 0$.

11(c). $F^{-1} = \left(\frac{(9 - u - 2v)}{2}, 5 - u - v\right)$. This is a diffeomorphism.

7. $GF = (g_1(f_1, f_2), g_2(f_1, f_2))$.

12(a). $GF = (g_1(f_1, \ldots, f_m), \ldots, g_p(f_1, \ldots, f_m))$

12(b). $(GF)_*(\alpha') = ((GF)(\alpha))'$, by Theorem I.7.8,

$=(G(F(\alpha)))'$

$=G_*((F(\alpha))')$

$=G_*F_*(\alpha')$.

12(c). Since any $v_p = \alpha'(0)$ for some α it follows that $(GF)_* = G_*F_*$.

DIFFERENTIAL GEOMETRY

I **Calculus on Euclidean Space**

II **Frame Fields**

III **Euclidean Geometry**

IV **Calculus on a Surface**

V **Shape Operators**

VI **Geometry of Surfaces in E^3**